Causal perturbation theory in terms of retarded products, and a proof of the Action Ward Identity

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Abstract

In the framework of perturbative algebraic quantum field theory a local construction of interacting fields in terms of retarded products is performed, based on earlier work of Steinmann [42]. In our formalism the entries of the retarded products are local functionals of the off shell classical fields, and we prove that the interacting fields depend only on the action and not on terms in the Lagrangian which are total derivatives, thus providing a proof of Stora's 'Action Ward Identity' [45]. The theory depends on free parameters which flow under the renormalization group. This flow can be derived in our local

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framework independently of the infrared behavior, as was first established by Hollands and Wald [32]. We explicitly compute non-trivial examples for the renormalization of the interaction and the field.

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Contents

1 Introduction

Among the various, essentially equivalent formulations of quantum field theory (QFT) the algebraic formulation [28, 27] seems to be the most appealing one from the conceptual point of view, but on the other side the least accessible one from the computational point of view. In perturbation theory, which is still the most successful method for making contact between theory and experiment, the approach towards a construction of interacting fields in terms of operators on a Hilbert space was popular in the early years, see e.g. [34]. But later it was largely abandoned in favor of a direct determination of Green functions. This is partially due to some complications which arise in the perturbative expansion of Wightman functions compared to the time ordered functions. Moreover, if the ultimate goal of QFT is the computation of the S-matrix, the approach via time ordered functions is more direct, in view of the LSZ-formulas, and the approach via Wightman functions seems to be a detour, which is conceptually nice but unimportant for practitioners.

There are, however, several reasons for a revision of this prevailing attitude. One is the desire to understand QFT on generic curved backgrounds. There, no general asymptotic condition à la LSZ exists; moreover, even the concepts of vacuum and particles loose their distinguished meaning, and one is forced to base the theory on the algebra of quantum fields (see e.g. [7] and references cited therein). Another reason is the connection to the classical limit. The algebra of perturbative quantum fields may be understood in terms of deformation quantization of the underlying Poisson algebra of free classical fields [13, 14], and one may hope that deformation quantization applies also to non-perturbative fields. But even in the traditional QFT on Minkowski space the algebraic formulation has great advantages because it completely separates the UV-problem from the IR-problem and, in principle, allows a consistent treatment of situations (like in QED [12]) where an S-matrix, strictly speaking, does not exist.

Actually, the perturbative construction of the algebras of quantum fields is possible, e.g. using the Bogoliubov-Epstein-Glaser approach of causal perturbation theory [3, 18]. There, the system of time ordered products of Wick polynomials of free quantum fields is recursively constructed, and interacting fields are given in terms of Bogoliubov's formula [3]

$$A_{\int g\mathcal{L}}(x) \stackrel{\mathrm{def}}{=} \frac{1}{i}\frac{\delta}{\delta h(x)}\Big(T\, e^{i\int g\mathcal{L}}\Big)^{-1}\Big(T\, e^{i\int g\mathcal{L}+hA}\Big)|_{h=0}\ , \tag{1.1}$$

where g and h are test functions and A is a polynomial in the basic fields and their partial derivatives.

The Taylor series expansion of the interacting field with respect to the interaction defines the retarded products

$$R_{n,1}(\mathcal{L}(x_1),\ldots,\mathcal{L}(x_n),A(x)) = \frac{\delta^n}{\delta g(x_1)\cdots\delta g(x_n)} A_{\int g\mathcal{L}}(x) \tag{1.2}$$

which can, by Bogoliubov's formula, be expressed in terms of time ordered and anti time ordered products. It is desirable, however, to have a direct construction of the retarded products, without the detour via the time ordered products. In particular, the approach to the classical limit simplifies enormously, since the retarded products are power series in $\hbar$, whereas the time ordered products are Laurent series (see the simplifications in [14] compared to [13]).

Such a construction was performed by Steinmann [42] (see also his recent book on QED [43]). We review his construction with some modifications. The most important one is the restriction to localized interactions (the support of the coupling function g in (1.1) is bounded) as it is characteristic for causal perturbation theory in the sense of Bogoliubov [3], Epstein and Glaser [18]. Therefore we don't have to discuss the asymptotic structure. Instead we use the algebraic adiabatic limit introduced in [7]. This limit relies on the observation that the algebraic structure of observables localized in a certain region does not depend on the behavior of the interaction outside of the region. This allows a construction of interacting fields but not of the S-matrix (Sect. 5). In this way we obtain a unified theory of massive and massless theories as well as of theories on curved space time (the latter are not discussed in the present paper).

We allow as interaction also terms with derivatives. Traditionally, in causal perturbation theory one uses, as arguments of time ordered and retarded products, on shell fields, i.e. fields which are subject to the free field equations[1] [18, 40, 16]. The inclusion of derivative couplings then leads to complications, and a consistent treatment requires a somewhat involved formalism [16]. Moreover, a change of the splitting between free and interaction terms in the Lagrangean requires a major effort. We therefore prefer to use off shell fields, thereby following a suggestion of Stora [45]. A natural question is

[1] An exception is Sect. 4 of the Epstein Glaser paper [18]: to interprete it consistently the arguments of time ordered products must be off shell fields.

then whether in this framework derivatives commute with time ordered and retarded products. This amounts to the problem whether the interacting fields depend only on the *action* S and not on how it is written as an integral $S = \int dx\, g(x)\mathcal{L}(x)$, i.e. whether total derivatives in $g\mathcal{L}$ can be ignored. The corresponding Ward identity was termed Action Ward Identity by Stora [45]. We will show in this paper that the Action Ward Identity can indeed be fulfilled.

We even go a step further: also the *values* of our retarded products are off shell fields, in contrast with the literature. For the physical predictions this makes no difference, and we gain technical simplifications.

The dependence of the theory on the renormalization conditions is usually analyzed in the adiabatic limit. In the algebraic adiabatic limit we can perform this analysis completely locally, thus avoiding all infrared problems (Sect. 5). Actually, such an analysis was already given by Hollands and Wald [32] for theories on curved space-times where the traditional adiabatic limit makes no sense, in general. In case of the scaling transformations we illustrate the formalism by computing (to lowest non-trivial order) examples for the renormalization of the interaction and the field.

Hollands and Wald [31] also introduce a new concept of scaling transformation which applies to fields on generic curved space-times. This concept already entails important consequences for massive theories on Minkowski space. We therefore adopt this point of view: among the axioms of causal perturbation theory we require smooth mass dependence[2] for $m \geq 0$ and almost homogeneous scaling. These conditions ensure that renormalization depends only on the short distance behavior of the theory, in agreement with the principle of locality.

A main question in perturbative QFT is whether symmetry with respect to a certain group G can be maintained in the process of renormalization. In Appendix C we prove that this is possible if all finite dimensional representations of G are completely reducible. In case of compact groups and for Lorentz-invariance we complete this existence result by giving a construction of a symmetric renormalization.

[2]In even dimensions this cannot be satisfied if the $*$-product is defined with respect to the usual two-point function of the free field; a modification of the $*$-product is necessary.

2 Axioms for retarded products

We consider for notational simplicity the theory of a real scalar field on d dimensional Minkowski space $\mathbb{M}$, $d > 2$. The classical configuration space $\mathcal{C}$ is the space $\mathcal{C}^\infty(\mathbb{M},\mathbb{R})$ and the field φ is the evaluation functional on this space: $(\partial^a\varphi)(x)(h) = \partial^a h(x)$, $a \in \mathbb{N}_0^d$. Let $\mathcal{F}$ be the set of all functionals F on $\mathcal{C}$ with values in the formal power series in $\hbar$ and which have the form

$$F(\varphi) = \sum_{n=0}^{N} \int dx_1 \dots dx_n\, \varphi(x_1)\cdots\varphi(x_n) f_n(x_1,\dots,x_n)\ , \qquad N < \infty\ , \tag{2.1}$$

where the f_n's are $\mathbb{C}[[\hbar]]$-valued distributions with compact support, which are symmetric under permutations of the arguments and whose wave front sets satisfy the condition

$$\mathrm{WF}(f_n) \cap \big(\mathbb{M}^n \times (\overline{V_+}^n \cup \overline{V_-}^n)\big) = \emptyset \tag{2.2}$$

and $f_0 \in \mathbb{C}[[\hbar]]$ (see Sect. 5.1 of [13] and Sect. 4 of [14]). The value of the functional $F(\varphi)$ for the argument $h \in \mathcal{C}$ is obtained by substituting everywhere h for φ on the right side of (2.1): $F(\varphi)(h) = F(h)$. An important example is

$$f_n(x_1,...,x_n) = \begin{cases} 0 & \text{for } \ n \neq k \\ \int dx\, f(x)\, \prod_{j=1}^k \partial^{a_j}\delta(x_j - x) & \text{for } \ n = k \end{cases} \tag{2.3}$$

(where $f \in \mathcal{D}(\mathbb{M})$ and $a_j \in \mathbb{N}_0^d$), which gives $F(\varphi) = (-1)^{\sum_j |a_j|}(\prod_{j=1}^k \partial^{a_j}\varphi)(f)$. $\mathcal{F}$ is a *-algebra with the classical product $(F_1 \cdot F_2)(h) := F_1(h) \cdot F_2(h)$, where F^* is obtained from F (2.1) by complex conjugation of all f_n's.

We introduce the functional

$$\omega_0 : \begin{cases} \mathcal{F} & \longrightarrow \quad \mathbb{C}[[\hbar]] \\ F & \mapsto \quad F(0) \equiv f_0 \end{cases} \tag{2.4}$$

which will be interpreted as 'vacuum state'. $\mathcal{F}(\mathcal{O})$ denotes the space of functionals localized in the spacetime region $\mathcal{O}$, i.e. which depend only on $\varphi(x)$ for $x \in \mathcal{O}$,

$$\mathcal{F}(\mathcal{O}) = \{F \in \mathcal{F} \quad | \quad \mathrm{supp}\, \frac{\delta F}{\delta\varphi} \subset \mathcal{O}\}\ .$$

Here, the functional derivatives of a polynomial functional F (2.1) are $\mathcal{F}$-valued distributions given by

$$\frac{\delta^k F}{\delta\varphi(x_1)\ldots\delta\varphi(x_k)} =$$
$$\sum_{n=k}^{N} \frac{n!}{(n-k)!} \int dy_1 \ldots dy_{n-k}\, \varphi(y_1)\cdots\varphi(y_{n-k}) f_n(x_1,\ldots,x_k,y_1,\ldots,y_{n-k}) \ . \tag{2.5}$$

Let $\Delta_+^{(m)}$ be the 2-point function of the free scalar field with mass m. On $\mathcal{F}$ we define an m-dependent associative product by

$$(F \star_m G)(\varphi) = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \int dx_1 \ldots dx_n dy_1 \ldots dy_n \frac{\delta^n F}{\delta\varphi(x_1)\cdots\delta\varphi(x_n)}$$
$$\cdot \prod_{i=1}^{n} \Delta_+^{(m)}(x_i - y_i) \frac{\delta^n G}{\delta\varphi(y_1)\cdots\delta\varphi(y_n)} \ , \tag{2.6}$$

which induces a product $\star_m : \mathcal{F}(\mathcal{O}) \times \mathcal{F}(\mathcal{O}) \to \mathcal{F}(\mathcal{O})$. The condition (2.2) on the coefficients f_n guarantees that the product is well defined, i.e. the pointwise product of distributions on the right side of (2.6) exists and the 'coefficients' of $(F \star_m G)$ satisfy again (2.2). $\star_m$ corresponds to a $\star$-product in the sense of deformation quantization [1] (see also [29, 5]), and it may be interpreted as Wick's Theorem for 'off-shell fields' (i.e. fields which are not restricted by any field equation). The corresponding algebras are denoted by $\mathcal{A}^{(m)}$ and $\mathcal{A}^{(m)}(\mathcal{O})$, respectively. For $\hbar = 0$ the product reduces to the classical product. Since we understand the functionals F, G and their product $(F \star_m G)$ as formal power series in $\hbar$, each equation must hold individually in each order of $\hbar$, in particular renormalization (see Sect. 3) has to be done in this sense.

The algebra of Wick polynomials is obtained by dividing out the ideal $\mathcal{J}^{(m)}$ generated by the field equation

$$\mathcal{J}^{(m)} = \{F(\varphi) = \sum_{n=1}^{N} \int dx_1 \ldots dx_n \varphi(x_1)\cdots\varphi(x_n)(\Box_{x_1} + m^2) f_n(x_1,\ldots x_n) \mid$$
$$f_n \text{ as above}\} \ . \tag{2.7}$$

The quotient algebra $\mathcal{F}_0^{(m)} \equiv \mathcal{F}/\mathcal{J}^{(m)}$ can be, for each fixed value of $\hbar > 0$, faithfully represented on Fock space by identifying the classical product of

fields $\pi\Big(\int dx_1...dx_n\,\varphi(x_1)...\varphi(x_n)\,f_n(x_1,...,x_n)\Big)$ with the normally ordered product $\int dx_1...dx_n\;:\varphi(x_1)...\varphi(x_n):\;f_n(x_1,...,x_n)$, where π is the canonical surjection $\pi:\mathcal{F}\to\mathcal{F}_0^{(m)}$ (Theorem 4.1 in [14]). ω_0 (2.4) induces a state on $\mathcal{F}_0^{(m)}$ which corresponds to the Fock vacuum. $\mathcal{F}_0^{(m)}(\mathcal{O})$ denotes the image of $\mathcal{F}(\mathcal{O})$ under π. Let $\mathcal{C}_0^{(m)}\subset\mathcal{C}$ be the space of smooth solutions of the free field equation. Since $F\in\mathcal{J}^{(m)}\Leftrightarrow F|_{\mathcal{C}_0^{(m)}}=0$, the canonical surjection π can alternatively be viewed as the restriction of the functionals $F\in\mathcal{F}$ to $\mathcal{C}_0^{(m)}$ (for details see [15]).

We are particularly interested in local functionals. We call a functional $F\in\mathcal{F}$ *local* if

$$\frac{\delta^2F}{\delta\varphi(x)\delta\varphi(y)}=0\quad\text{ for }\quad x\neq y\ .$$

Local functionals are of the form

$$F=\int dx\,\sum_{i=1}^N A_i(x)h_i(x)\equiv\sum_{i=1}^N A_i(h_i)\tag{2.8}$$

where the A_i's are polynomials of the field φ and its derivatives and the h_i's are test functions with compact support, $h_i\in\mathcal{D}(\mathbb{M})$. The set of local functionals will be denoted by $\mathcal{F}_{\mathrm{loc}}$.

Remark: There exist faithful Hilbert space representations of the off-shell fields. For example, a faithful representation π of the algebra $\mathcal{F}$ (with the classical product) is obtained by interpreting $\mathcal{F}$ as a vector space and the representation is defined by left-multiplication: $\pi(F)\,G:=FG$ $(F,G\in\mathcal{F})$. A possible scalar product reads

$$\langle F,G\rangle:=\omega_0(F^*\star_g G)\ ,\tag{2.9}$$

where $\star_g$ is the $\star$-product (2.6) with $\Delta_+^{(m)}$ replaced by the 2-point function $\Delta_+^{[g]}$ of a generalized free field with weight function $g\in\mathcal{D}(\mathbb{R}_+)$, i.e. $\Delta_+^{[g]}(y)=\int dm^2\,g(m^2)\,\Delta_+^{(m)}(y)$. (Note that the smoothness of g excludes the case of a free field, $\Delta_+^{[g]}=\Delta_+^{(m_0)}$ for some m_0, in which (2.9) would be degenerate.)

Compared with their Hilbert space representations, the algebras $\mathcal{F}$ and $\mathcal{F}_0^{(m)}$ are more flexible and more convenient[3]; and, as it is demostrated by this paper, they provide all necessary information.

[3]For example one does not need to care about domains of unbounded operators.

We want to construct, for any pair of local functionals $F, G \sim \hbar^0$ the quantum field theoretical operator $F_{G/\hbar}$ ("interacting field") which corresponds to F under the interaction term[4] $G/\hbar$. F_G should be a *formal power series* in G where each term is an element of $\mathcal{F}(\mathcal{O})$ if $F, G \in \mathcal{F}(\mathcal{O})$. Here we deviate essentially from the usual formalism of perturbative QFT: there the interacting fields are Fock space operators (which means in our algebraic formulation that they are elements of $\mathcal{F}_0^{(m)}$). Motivated by the study of the Peierls bracket [36, 15], we define them to be unrestricted functionals ('off-shell fields'). This simplifies strongly the proof of the 'Main Theorem of perturbative renormalization' (Sect. 4.2) and e.g. the formulation of the renormalization conditions 'Covariance' and 'Field Independence' given below. At $\hbar = 0$, the restriction $F_G|_{\mathcal{C}_0^{(m)}}$ is the (perturbative) classical retarded field as constructed in [15] (see also [11, 36]).

We require the following properties, which may be motivated by their validity in classical field theory (see [15]):

Initial condition: For $G = 0$ we obtain the original functional,

$$F_0 = F \ .$$

Causality: Fields are not influenced by interactions which take place later:

$$F_{G+H} = F_G$$

if there is a Cauchy surface such that F is localized in its past and H in its future.

GLZ Relation: The Poisson bracket $\{F_{G/\hbar}, H_{G/\hbar}\} \stackrel{\text{def}}{=} \frac{i}{\hbar}[F_{G/\hbar}, H_{G/\hbar}]_{\star m}$ satisfies the GLZ relation [23, 42, 15]

$$\{F_{G/\hbar}, H_{G/\hbar}\} = \frac{d}{d\lambda}(F_{(G+\lambda H)/\hbar} - H_{(G+\lambda F)/\hbar})|_{\lambda=0} \ .$$

Due to the GLZ relation and (2.6) the interacting fields depend on the $*$-product (2.6) and with that they depend on the mass m of the free field.

Steinmann [42] discovered that by these conditions an inductive construction of the perturbative expansion of F_G (i.e. of the retarded products (1.2)) can be done up to local functionals which could be added in every order.

[4] Mostly we will set $\hbar = 1$.

But these undetermined terms correspond to the renormalization ambiguities which are there anyhow in perturbative quantum field theory. One may reduce these ambiguities by prescribing normalization conditions which are satisfied in classical field theory [15]. We impose the following conditions:

Unitarity: Complex conjugation induces an involution $F \mapsto F^*$ of the algebra which after restriction to $\mathcal{C}_0^{(m)}$ becomes the formal adjoint operation on Fock space. We require

$$(F_G)^* = F^*_{G^*} \ .$$

This condition implies that a real interaction G leads formally to a unitary S-matrix and hermitian interacting fields (if $F^* = F$).

Covariance: The Poincaré group $\mathcal{P}_+^\uparrow$ has a natural automorphic action β on $\mathcal{F}$. We require

$$\beta_L(F_G) = \beta_L(F)_{\beta_L(G)} \quad \forall L \in \mathcal{P}_+^\uparrow \ .$$

(See [8] and [31] for the formulation of covariance on curved spacetime.) In addition, **global inner symmetries** (in our case the field parity $\alpha : \varphi \mapsto -\varphi$) should be preserved,

$$\alpha(F_G) = \alpha(F)_{\alpha(G)} \ . \tag{2.10}$$

Field Independence: A coherent prescription for the renormalization of polynomials in the basic fields and all sub-polynomials can be obtained by the following condition

$$\langle h, \frac{\delta}{\delta\varphi} F_G\rangle = \langle h, \frac{\delta F}{\delta\varphi}\rangle_G + \frac{d}{d\lambda}|_{\lambda=0} F_{G+\lambda\langle h,\frac{\delta G}{\delta\varphi}\rangle} \ \ , \ h \in \mathcal{D}(\mathbb{M}) \tag{2.11}$$

(where $\langle h, \frac{\delta H}{\delta\varphi}\rangle \equiv \int dx\, h(x) \frac{\delta H(x)}{\delta\varphi(x)}$ for $H \in \mathcal{F}$). Below it will turn out that this condition is equivalent to the natural generalization to our 'off-shell formalism' of the causal Wick expansion given in Sect. 4 of [18]. (The latter is equivalent to the condition **N3** in [12], which is also called 'relation to time-ordered products of sub-polynomials' in [16].)

Field equation: The renormalization ambiguities can be used to fulfill the Yang-Feldman equation 'off-shell':

$$\varphi_G(x) = \varphi(x) - \int dy\, \Delta_m^{\rm ret}(x-y)\Big(\frac{\delta G}{\delta\varphi(y)}\Big)_G \ . \tag{2.12}$$

The latter condition may be enforced by requiring the validity of all local identities which hold classically as a consequence of the field equation. This was termed Master Ward Identity in [15]. Since the Master Ward Identity cannot always be fulfilled we do not impose it here.

Smoothness in the mass $m \geq 0$: the classical interacting fields depend smoothly on the mass of the free fields in the range $m \geq 0$. In the quantum case, in even dimensional spacetime, this is no longer true even for the free fields because of logarithmic singularities of the two-point function Δ_m^+ (see Appendix A).[5] One can remedy this defect by passing to an equivalent star product $\star_{m,\mu}$ which is defined by (2.6) with $\Delta_m^{+(d)}$ replaced by

$$H_m^{\mu\,(d)}(x) \equiv \Delta_m^+(x) - \log(m^2/\mu^2)\,\gamma(x)\ , \quad \gamma(x) \equiv m^{d-2}\,h^{(d)}(m^2x^2)\ . \tag{2.13}$$

(This procedure follows essentially [30].) $\mu > 0$ is an additional mass parameter. $h^{(d)}$ is analytic and it is chosen such that $H_m^{\mu\,(d)}$ is smooth in $m \geq 0$. Explicitly we choose $h^{(2l+1)} \equiv 0$ and $h^{(4+2k)}(y) \equiv \pi^{-k}\,f^{(k)}(y)$, where f is given in Appendix A by (A.9). With this particular choice $H_m^{\mu\,(d)}$ solves the Klein-Gordon equation, but we point out that this is not necessary for the purposes of this paper,[6] cf. [30]. To show that the product $\star_{m,\mu}$ is equivalent to $\star_m$, we introduce the transformation

$$\Big(\frac{m}{\mu}\Big)^{\Gamma} = 1 + \sum_{k=1}^{\infty} \frac{1}{k!}\Big(\log(m/\mu)\cdot\Gamma\Big)^k \tag{2.14}$$

where Γ is the operator

$$\Gamma \equiv \Gamma^{(m)} \equiv \int dx\,dy\,\gamma(x-y)\,\frac{\delta^2}{\delta\varphi(x)\delta\varphi(y)}\ . \tag{2.15}$$

By using

$$e^{\lambda\varphi(f)} \star_m e^{\lambda\varphi(g)} = e^{\lambda\varphi(f+g)}\; e^{\lambda^2\,(f,\Delta_m^+ g)} \tag{2.16}$$

[5]We are indebted to Stefan Hollands for pointing out to us that this fact invalidates our treatment of the scaling behavior in an older version of this manuscript.

[6]In order that $\star_{m,\mu}$ induces a well defined $\star$-product on $\mathcal{F}_0^{(m)} \equiv \mathcal{F}/\mathcal{J}^{(m)}$ it is needed that $H_m^{\mu\,(d)}$ solves the Klein-Gordon equation. However, we always work with off-shell fields (i.e. with $\mathcal{F}$).

(understood as formal power series in λ), the same formula for $(\star_{m,\mu}, H_m^\mu)$ and

$$\left(\frac{m}{\mu}\right)^\Gamma e^{\lambda\varphi(f)} = e^{\lambda\varphi(f)} \left(\frac{m}{\mu}\right)^{\lambda^2 \int dx\, dy\ \gamma(x-y)\, f(x)\, f(y)} , \tag{2.17}$$

we find that $(\frac{m}{\mu})^\Gamma$ intertwines between $\star_{m,\mu}$ and $\star_m$:

$$\left(\frac{m}{\mu}\right)^\Gamma \left(F \star_{m,\mu} G\right) = \left(\frac{m}{\mu}\right)^\Gamma (F) \star_m \left(\frac{m}{\mu}\right)^\Gamma (G) , \quad F, G \in \mathcal{F} . \tag{2.18}$$

As mentioned after (2.7), the normally ordered product $:\varphi(f)^n:_m$ (where normal ordering is done with Δ_m^+) agrees with $\pi((\varphi(f))^n)$. The modified normally ordered product $:\varphi(f)^n:_{m,\mu}$ (i.e. normal ordering is done with H_m^μ) agrees with $\pi((\frac{m}{\mu})^{-\Gamma}\,(\varphi(f))^n)$. With that (2.17) yields that the two different kinds of Wick powers are related by

$$: e^{\lambda\,\varphi(x)} :_{m,\mu} =: e^{\lambda\,\varphi(x)} :_m \left(\frac{m}{\mu}\right)^{-\lambda^2\,\gamma(0)} . \tag{2.19}$$

We may now introduce interacting fields with respect to the modified $*$-product,

$$(F_G)^{(m,\mu)} := \left(\frac{m}{\mu}\right)^{-\Gamma} \left(((\frac{m}{\mu})^\Gamma (F))^{(m)}_{(\frac{m}{\mu})^\Gamma (G)} \right) , \tag{2.20}$$

where $F_G^{(m)}$ denotes the interacting field with respect to the usual $*$-product (2.6). Note that at $m = 0$ the two kinds of interacting fields agree: $F_G^{(0,\mu)} = F_G^{(0)}$. Since Γ is local (i.e. supp $\frac{\delta(\Gamma F)}{\delta\varphi} \subset$ supp $\frac{\delta F}{\delta\varphi}$), field independent (in the sense that γ is field independent), Poincaré invariant and commutes with the $*$-operation (since γ is Poincaré invariant and real), the modified interacting fields satisfy the same conditions as the original interacting fields (where, of course, in the GLZ relation the commutator with respect to the modified $*$-product has to be used).

Our smoothness condition now takes the following form: we require that the maps

$$m \mapsto (F_G)^{(m,\mu)} , \qquad F, G \in \mathcal{F}_{\mathrm{loc}} , \quad \mu > 0 , \tag{2.21}$$

are smooth (in the sense of one sided derivatives at $m=0$).[7]

Remarks: (1) The fact that the $\star$-products $\star_{m,\mu}$ are equivalent for different values of μ (which follows immediately from (2.18)) may also be formulated in the following way. Introduce new fields by

$$\varphi^{\otimes n}(x_1,\dots,x_n)_{m,\mu} = \Big(\frac{m}{\mu}\Big)^{-\Gamma} \varphi(x_1)\dots\varphi(x_n)\ .$$

Every functional $F\in\mathcal{F}$ may be expanded in any of these fields

$$F=\sum_n \int dx_1...dx_n\, f_n^{m,\mu}(x_1,\dots,x_n)\,\varphi^{\otimes n}(x_1,\dots,x_n)_{m,\mu}$$

with suitable coefficients $f_n^{m,\mu}$. The different $\star$-products then arise when the $\star$-product $\star_m$ is expressed in terms of the coefficients:

$$\begin{aligned} F\star_m G &= \sum_{n,k}\int dx_1dy_1...\, f_n^{m,\mu}(x_1,...)\, g_k^{m,\mu}(y_1,...)\, \Big(\frac{m}{\mu}\Big)^{-\Gamma} \Big(\varphi(x_1)...\star_{m,\mu}\varphi(y_1)...\Big) \\ &= \sum\int f^{m,\mu}\cdot (\prod H_m^\mu)\cdot g^{m,\mu}\cdot(\varphi^{\otimes})_{m,\mu}\ . \end{aligned}$$

Hence, the choice of μ can be understood as the choice of a basis for $\mathcal{A}^{(m)}\equiv(\mathcal{F},\star_m)$.
(2) The introduction of the modified $*$-product and modified interacting fields (2.20) can be avoided by requiring that the function $m\mapsto F_G^{(m)}$ is *almost smooth for* $m\downarrow 0$ for all $F;G\in\mathcal{F}_{\rm loc}$. In doing so a function $\mathbb{R}_+\ni m\mapsto f(m)$ is called almost smooth for $m\downarrow 0$, if for any fixed mass parameter $\mu>0$ there exist polynomials $p_{k,\mu}$, $k\in\mathbb{N}_0$, such that for each $n\in\mathbb{N}_0$ it holds

$$m^{-n}\Big(f(m)-\sum_{k\le n} m^k\, p_{k,\mu}(\log\frac{m}{\mu})\Big)\longrightarrow 0 \tag{2.22}$$

for $m\downarrow 0$. Note that the polynomials $p_{k,\mu}$ are *uniquely* determined by this condition. Then the scaling expansion (3.10) has to be generalized correspondingly. We do not go this way, because the treatment

[7] Due to (2.13) the interacting fields depend only on m^2, and setting $(F_G)^{(-m,\mu)} := (F_G)^{(m,\mu)}$ the map (2.21) can be extended to $m<0$. It is even possible to require that $(F_G)^{(m,\mu)}$ is smooth in m^2 (which is a stronger condition than smoothness in m). However, note that this footnote is not valid for spinor fields.

of the renormalization ambiguities is much simpler for the modified interacting fields: the map $D_H^{(m)}$ of the Main Theorem (i.e. Theorem 8), which gives a finite renormalization of the *modified* interacting fields, is free of $(\log m)$-terms (4.14); but the corresponding $D^{(m)}$ of the original interacting fields contains such terms (4.15).

Scaling: Under a simultaneous scaling of the coordinates and the mass, $(x, m) \mapsto (\rho x, \rho^{-1} m)$, the interacting classical fields transform homogeneously. This can no longer be maintained for the quantized theory, together with the requirement of smoothness at $m = 0$, even for free fields (in even dimensions) since H_m^μ does not scale homogeneously (A.12). However, we will show that the retarded products can be normalized such that they scale almost homogeneously (i.e. up to logarithmic terms).

The last two normalization conditions were first imposed in [31] in the more general context of renormalization on curved spacetime. In the traditional literature (e.g. [18, 42, 7]) instead the weaker requirement was used that 'renormalization may not make the interacting fields more singular' (in the UV-region), see footnote 13.

The listed conditions can be translated in a straightforward way into conditions on the **retarded products** $R_{n,1}$ which are by definition (1.2) the Taylor coefficients of the interacting field with respect to the interaction,

$$A(f)_{\mathcal{L}(g)} = \sum_{n=0}^{\infty} \frac{1}{n!} R_{n,1}\big(\mathcal{L}(g)^{\otimes n}, A(f)\big) \equiv R(e_\otimes^{\mathcal{L}(g)}, A(f)) \tag{2.23}$$

with $g, f \in \mathcal{D}(\mathbb{M})$ and $A, \mathcal{L} \in \mathcal{P}$, where $\mathcal{P}$ is the algebra of polynomials in the classical field φ and its partial derivatives (with respect to pointwise multiplication). We use (2.23) for both kinds of interacting fields, $F_G^{(m)}$ and $F_G^{(m,\mu)}$, and by writing R (or $R_{n,1}$) we mean both kinds of retarded products, $R^{(m)}$ and $R^{(m,\mu)}$. The retarded product $R_{n-1,1}$ is a **linear** map, from $\mathcal{F}_{\mathrm{loc}}^{\otimes n}$ into $\mathcal{F}$ which is symmetric in the first $n-1$ variables. In the last expression of (2.23) the sequence $R \equiv (R_{n-1,1})_{n\in\mathbb{N}}$ is viewed as a map

$$R : T\mathcal{F}_{\mathrm{loc}} \longrightarrow \mathcal{F} \ , \quad \text{where} \quad T\mathcal{F}_{\mathrm{loc}} \stackrel{\mathrm{def}}{=} \oplus_{n=0}^{\infty} \mathcal{F}_{\mathrm{loc}}^{\otimes n} \tag{2.24}$$

and $R_{-1,1} \stackrel{\mathrm{def}}{=} 0$ which is extended by linearity to formal power series. It is sometimes advantageous to interprete $R_{n-1,1}$ as $\mathcal{F}$-valued distribution in n

variables on the test function space $\mathcal{D}(\mathbb{M},\mathcal{P})$ (cf. [15]). In particular our retarded products are multi-linear in the fields $A \in \mathcal{P}$.

We also use the symbolic notation $R_{n-1,1}(x_1,\ldots,x_n)$ for a distribution which takes values in $\mathcal{F}\otimes\mathcal{P}'_n$, where $\mathcal{P}'_n$ is the dual vector space of $\mathcal{P}^{\otimes n}$. After insertion of fields $A_1,\ldots,A_n \in \mathcal{P}$ we obtain $\mathcal{F}$-valued distributions on $\mathcal{D}(\mathbb{M}^n)$ which are symbolically written as $R_{n-1,1}(A_1(x_1),\ldots,A_n(x_n))$.

Action Ward Identity (AWI): Since the retarded products depend only on the functionals, derivatives may be shifted from the test functions to the fields and vice versa. Hence, the associated distributions must satisfy the Action Ward Identity

$$\partial^x_\mu R_{n-1,1}(\ldots A_k(x)\ldots) = R_{n-1,1}(\ldots,\partial_\mu A_k(x),\ldots)\ . \tag{2.25}$$

Symmetry: Motivated by (2.23) we require that $R_{n-1,1}$ is symmetric in the first $(n-1)$ factors. This property is reflected in the convention for the lower indices of $R_{n-1,1}$.

Initial condition: $\quad R_{0,1}(F) = F\ .$

Causality:

$$\mathrm{supp}\, R_{n-1,1} \subset \{(x_1,\ldots,x_n)\in\mathbb{M}^n \mid x_i \in x_n + \overline{V}_-,\ \forall i = 1,\ldots n-1\}\ . \tag{2.26}$$

GLZ relation:

$$R_{n-1,1}(\ldots,y,z) - R_{n-1,1}(\ldots,z,y) = \hbar\, J_{n-2,2}(\ldots,y,z) \tag{2.27}$$

where $J_{n-2,2}$ is an algebra valued distribution in n variables, $n\geq 2$, which is defined by

$$J_{n-2,2}(A_1(x_1),\ldots,A_{n-2}(x_{n-2}),B(y),C(z)) \stackrel{\mathrm{def}}{=} \sum_{I\subset\{1,\ldots,n-2\}} \Big\{ R_{|I|,1}(A_i(x_i), i\in I, B(y)), R_{|I^c|,1}(A_i(x_i), i\in I^c, C(z)) \Big\}\ . \tag{2.28}$$

Here, I^c is the complement of I in $\{1,\ldots,n-2\}$. Obviously, $J_{n-2,2}$ is symmetric in the first $n-2$ variables and antisymmetric in the last 2

variables. The properties of $J_{n-2,2}$ stated in the following Lemma are necessary conditions for the GLZ-relation and the causality of $R_{n-1,1}$. But actually, they are already fulfilled as a consequence of the definition of J (2.28), and the GLZ relation and Causality of the retarded products to lower orders [42].

Lemma 1. *(a) J satisfies the Jacobi identity*

$$J_{n-2,2}(\dots,x,y,z)+\mathrm{cycl}(x,y,z)=0\ . \tag{2.29}$$

(b) The support of $J_{n-2,2}$ is contained in the set

$$\{x_i\in x_n+\bar{V}_-,i=1,\dots,n-1\}\cup\{x_i\in x_{n-1}+\bar{V}_-,i=1,\dots n-2,n\}\ . \tag{2.30}$$

Proof. (cf. [42]) We start with the Jacobi identity of the Poisson bracket and use the notation $x_M\equiv(x_m|m\in M)$, where $M\subset\{1,...,n-1\}$. So we know

$$\sum_{I\sqcup H\sqcup L=\{1,\dots,n-1\}}\{\{R(x_I,x),R(x_H,y)\},R(x_L,z)\}+\mathrm{cycl}(x,y,z)=0, \tag{2.31}$$

where $\sqcup$ means the disjoint union. The sum over all decompositions of $K\equiv I\sqcup H(=\text{fixed})$ of the inner Poisson bracket is equal to $J_{|K|,2}(x_K,x,y)$, $|K|\leq n-1$, which splits into $R(x_K,x,y)-R(x_K,y,x)$ due to the validity of the GLZ relation to lower orders. With that we obtain

$$\begin{aligned}0=&\sum_{K\sqcup L=\{1,\dots,n-1\}}\{(R(x_K,x,y)-R(x_K,y,x)),R(x_L,z)\}+\mathrm{cycl}(x,y,z)\\ &=J_{n,2}(x_1,\dots,x_{n-1},x,y,z)+\mathrm{cycl}(x,y,z).\end{aligned} \tag{2.32}$$

(b) By definition of J and the support properties of R it follows that $J_{n-2,2}(x_1,\dots,x_{n-2},y,z)$ vanishes if one of the first $n-2$ arguments is not in the past of $\{y,z\}$.

It remains to show that it vanishes also for $(y-z)^2<0$. If one of the first $n-2$ arguments is different from y and z, and is in the past of, say y, then by the Jacobi identity J has to vanish.

If, on the other hand, all arguments x_i are sufficiently near to either y or z, then they are space-like to the other point, hence all retarded

products in the definition of J vanish up to those where all arguments in the first factor are near to y and all arguments in the second factor are near to z. But then the Poisson bracket of these retarded products vanishes, since by assumption the retarded products are localized at their arguments.[8] $\square$

An immediate consequence of the Initial condition and the GLZ relation is

$$R_{n-1,1}(F_1, ..., F_n) = \mathcal{O}(\hbar^{(n-1)}) \quad \text{if} \quad F_1, ..., F_n \sim \hbar^0 \ . \tag{2.33}$$

Field Independence: The condition (2.11) translates into

$$\frac{\delta}{\delta\varphi(x)} R_{n-1,1}(F_1, \ldots, F_n) = \sum_{l=1}^{n} R_{n-1,1}(F_1, \ldots, \frac{\delta F_l}{\delta\varphi(x)}, \ldots, F_n) \ . \tag{2.34}$$

This condition determines the retarded product on the left side in terms of the retarded products on the right side, up to its value at $\varphi = 0$, i.e. its vacuum expectation value. Since by definition, the functionals F_i are polynomials in φ, one obtains the finite Taylor expansion[9]

$$\begin{aligned} R_{n-1,1}(F_1, \ldots, F_n) = \sum_{l_1 \ldots l_n} \frac{1}{l_1! \ldots l_n!} \int dx_{11} \ldots dx_{1l_1} \ldots dx_{n1} \ldots dx_{nl_n} \\ \omega_0\Big(R_{n-1,1}\big(\frac{\delta^{l_1} F_1}{\delta\varphi(x_{11}) \ldots \delta\varphi(x_{1l_1})}, \ldots, \frac{\delta^{l_n} F_n}{\delta\varphi(x_{n1}) \ldots \delta\varphi(x_{nl_n})}\big)\Big) \\ \varphi(x_{11}) \ldots \varphi(x_{1l_1}) \ldots \varphi(x_{n1}) \ldots \varphi(x_{nl_n}) \ , \end{aligned} \tag{2.35}$$

where the coefficients $\omega_0\big(R_{n-1,1}(\ldots)\big)$ are restricted by the other axioms. The retarded product on the right side is well-defined, because, due to $F_k \in \mathcal{F}_{\mathrm{loc}}$, the support of $\delta^l F_k/\delta\varphi(x_1) \ldots \delta\varphi(x_l)$ is contained in the total diagonal $x_1 = x_2 = \ldots = x_l$. After integrating out the corresponding δ-distributions the right side of (2.35) is a sum of terms

[8] The proof of the last fact given by Steinmann [42] is much more involved since he does not assume the localization property of the retarded products.

[9] In (2.35) and (2.36) the product of fields is the classical product, $\big(\varphi(x)\varphi(y)\big)(h) = h(x)h(y)$, and not the $\star$-product (2.6).

of the form

$$\int d^{nd}x\,\omega_0\Big(R_{n-1,1}\big(A_1(x_1),\ldots,A_n(x_n)\big)\Big)\prod_{i=1}^{n}h_i(x_i)\prod_{j_i=1}^{l_i}\partial^{a_{ij_i}}\varphi(x_i) \tag{2.36}$$

with $h_i\in\mathcal{D}(\mathbb{M})$, $A_i\in\mathcal{P}$ and multi-indices $a_{ij_i}\in\mathbb{N}_0^d$. Due to translation invariance of the vacuum, the coefficients in this expansion depend on the relative coordinates only. In particular, their wave front set satisfies the condition (2.2) on the admissible coefficients in $\mathcal{F}$.

The condition (2.34) that the retarded product is independent of φ, is the only axiom which relies on the fact that one perturbs around a theory with an action which is of second order in the field. Indeed, in the general case, the retarded products of classical field theory depend on the action only via its second functional derivative, see Proposition 1 of [15].

The restriction of (2.35) to $\mathcal{C}_0^{(m)}$ is the causal Wick expansion of Epstein and Glaser [18]. In particular the 'coefficients' $\omega_0(R_{n-1,1}(...))$ are exactly the same as in the on-shell formalism of [18].

Smoothness in the mass $m\geq 0$: In terms of the retarded products our requirement reads that

$$R^{(m,\mu)}_{n-1,1}(F_1,\ldots,F_n)=(\frac{m}{\mu})^{-\Gamma}\left(R^{(m)}_{n-1,1}\Big((\frac{m}{\mu})^{\Gamma}(F_1),\ldots,(\frac{m}{\mu})^{\Gamma}(F_n)\Big)\right) \tag{2.37}$$

depends smoothly on $m\;\forall m\geq 0$. By this we mean that $R^{(m,\mu)}_{n-1,1}(F_1,...,F_n)(h)$ is smooth as a function of m for all $F_1,...,F_n\in\mathcal{F}_{\mathrm{loc}}$ and all field configurations $h\in\mathcal{C}$, and that the derivatives are of the form $G(h)$ with $G\in\mathcal{F}$.[10] In particular this smoothness implies that $R^{(0,\mu)}_{n-1,1}$ can be obtained by the limit $\lim_{m\downarrow 0}R^{(m,\mu)}_{n-1,1}$ in the sense of distributions on $\mathcal{D}(\mathbb{M}^n)$.

Scaling: As an introduction and for later purpose we first define 'almost homogeneous scaling' for a distribution under rescaling of the coordinates (cf. [31]):

[10] The latter condition is necessary since the distributions occurring in the representation (2.1) of G have to satisfy the wave front set condition (2.2).

Definition 1. *A distribution $t \in \mathcal{D}'(\mathbb{R}^k)$ (or $\mathcal{D}'(\mathbb{R}^k \setminus \{0\})$) scales almost homogeneously with degree $D \in \mathbb{R}$ and power $N \in \mathbb{N}_0$ if*

$$(\sum_{r=1}^{k} z_r \partial_{z_r} + D)^{N+1} t(z_1, ..., z_k) = 0 \tag{2.38}$$

and N is the minimal natural number with this property. For $N = 0$ the scaling is homogeneous with degree D.

The condition (2.38) is equivalent to

$$0 = (\rho\partial_\rho)^{N+1}\Big(\rho^D t(\rho z_1, \ldots, \rho z_k)\Big) = \frac{\partial^{N+1}}{\partial(\log \rho)^{N+1}}\Big(\rho^D t(\rho z_1, \ldots, \rho z_k)\Big) \ . \tag{2.39}$$

Hence, t scales almost homogeneously with degree D and power N if $\rho^D t(\rho z_1, \ldots, \rho z_k)$ is a polynomial of $\log \rho$ with degree N.

To formulate almost homogeneous scaling for the retarded products under simultaneous rescalings of the coordinates and the mass m we introduce some tools. The *mass dimension of a monomial* in $\mathcal{P}$ is fixed by the conditions

$$\dim(\partial^a \varphi) = \frac{d-2}{2} + |a| \quad \text{and} \quad \dim(A_1 A_2) = \dim(A_1) + \dim(A_2) \tag{2.40}$$

for all monomials $A_1, A_2 \in \mathcal{P}$. This introduces a grading for $\mathcal{P}$:

$$\mathcal{P} = \oplus_j \mathcal{P}_j \ , \tag{2.41}$$

where $\mathcal{P}_j$ is the linear span of all monomials with mass dimension j. The mass dimension of $A = \sum_j A_j$, with $A_j \in \mathcal{P}_j$, is the maximum of the contributing j's. We also introduce the set of all field polynomials which are homogeneous in the mass dimension

$$\mathcal{P}_{\text{hom}} \stackrel{\text{def}}{=} \bigcup_j \mathcal{P}_j \ . \tag{2.42}$$

A scaling transformation σ_ρ is introduced (in analogy to [32]) as an automorphism of $\mathcal{F}$ (considered as an algebra with the classical product) by

$$\sigma_\rho(\varphi(x)) = \rho^{\frac{2-d}{2}} \varphi(\rho^{-1} x) \ . \tag{2.43}$$

Note $\omega_0 \circ \sigma_\rho = \omega_0$. Due to $\rho^{d-2}\Delta_+^{(\rho^{-1}m)}(\rho x) = \Delta_+^{(m)}(x)$, σ_ρ is also an algebra isomorphism from $\mathcal{A}^{(\rho^{-1}m)}$ to $\mathcal{A}^{(m)}$. However, denoting by $\mathcal{A}^{(m,\mu)}$ the algebra $(\mathcal{F}, \star_{m,\mu})$, σ_ρ is an isomorphism from $\mathcal{A}^{(\rho^{-1}m,\rho^{-1}\mu)}$ to $\mathcal{A}^{(m,\mu)}$, but *not* from $\mathcal{A}^{(\rho^{-1}m,\mu)}$ to $\mathcal{A}^{(m,\mu)}$, since $\rho^{d-2}\, H_{\rho^{-1}m}^{\rho^{-1}\mu\,(d)}(\rho x) = H_m^{\mu\,(d)}(x)$. For $m = 0$ the coordinates are scaled only and σ_ρ is an automorphism of $\mathcal{A}^{(m=0)} = \mathcal{A}^{(m=0,\mu)}$. For $A \in \mathcal{P}_{\mathrm{hom}}$, we obtain

$$\rho^{\dim(A)}\sigma_\rho(A(\rho x)) = A(x). \tag{2.44}$$

So, with the identification given by σ_ρ, they scale homogeneously with degree given by their mass dimension. By inserting the definitions one finds

$$\sigma_\rho \Gamma^{(\rho^{-1}m)}\sigma_\rho^{-1} = \Gamma^{(m)} \quad \text{and hence} \quad \sigma_\rho \Big(\frac{m}{\mu}\Big)^{\Gamma^{(\rho^{-1}m)}} \sigma_\rho^{-1} = \Big(\frac{m}{\mu}\Big)^{\Gamma^{(m)}} . \tag{2.45}$$

The property

$$\sigma_\rho(\sigma_\rho^{-1}F \star_0 \sigma_\rho^{-1}G) = F \star_0 G \tag{2.46}$$

of the $\star$-product at $m = 0$ cannot be maintained for the retarded products, i.e.

$$R^{(0)}_{n-1,1\,\rho} := \sigma_\rho \circ R^{(0)}_{n-1,1} \circ (\sigma_\rho^{-1})^{\otimes n} \tag{2.47}$$

will differ from $R^{(0)}_{n-1,1}$, in general. But one can reach that $R^{(0)}_{n-1,1}$ *scales almost homogeneously with degree zero*, i.e. $R^{(0)}_{n-1,1\,\rho}$ has polynomial behavior in $\log\rho$, in the sense that for all $F_1, ..., F_n \in \mathcal{F}_{\mathrm{loc}}$, $R^{(0)}_{n-1,1\,\rho}(F_1, ..., F_n)$ is a polynomial in $\log\rho$. In the *massive case*, scaling relates retarded products for different masses; therefore, the condition of homogeneity gives no restriction at a fixed mass. Our condition of almost homogeneous scaling states that

$$R^{(m)}_{n-1,1\,\rho} := \sigma_\rho \circ R^{(\rho^{-1}m)}_{n-1,1} \circ (\sigma_\rho^{-1})^{\otimes n}\ , \tag{2.48}$$

or equivalently

$$R^{(m,\mu)}_{n-1,1\,\rho} := \sigma_\rho \circ R^{(\rho^{-1}m,\mu)}_{n-1,1} \circ (\sigma_\rho^{-1})^{\otimes n} = \Big(\frac{m}{\rho\mu}\Big)^{-\Gamma^{(m)}} \circ R^{(m)}_{n-1,1\,\rho} \circ \Big(\Big(\frac{m}{\rho\mu}\Big)^{\Gamma^{(m)}}\Big)^{\otimes n} \tag{2.49}$$

(where (2.45) is used), has polynomial behavior in $\log\rho$. We will see, that this condition together with the smoothness requirement, imposes non-trivial restrictions also in the massive case. As mentioned above, in even dimensions, the smoothness condition requires the transition to an equivalent $*$-product which depends on an additional mass parameter μ.

It is obvious how the remaining conditions on the interacting fields (**Unitarity**, **Covariance** and **Field equation**) read in terms of the retarded products. With regard to the Field equation note that the retarded propagator is the same for the $R^{(m)}$-products and the $R^{(m,\mu)}$-products:

$$R^{(m,\mu)}_{1,1}(\varphi(y),\varphi(x)) = \Delta_m(x-y)\,\Theta(x^0-y^0) = \Delta^{\rm ret}_m(x-y) = R^{(m)}_{1,1}(\varphi(y),\varphi(x))\ . \tag{2.50}$$

Hence, the Yang-Feldman equation (2.12) has precisely the same form for both kinds of interacting fields.

From Bogoliubov's definition of interacting fields (1.1) it follows that there is a unique correspondence between retarded products and *time ordered products*, see e.g. [18]. So the axioms given in this section can equivalently be formulated in terms of time ordered products. This is done in Appendix E. In addition we give there an explicit formula which describes the time ordered products in terms of retarded products.

3 Construction of the retarded products

Our procedure is based on the strategies developed in [42, 44] and [7].

3.1 Inductive step outside of the total diagonal

If the retarded products with less than n factors are given, we can define the distribution $J_{n-2,2}$. But, by the GLZ relation, Symmetry and Causality, $R_{n-1,1}$ *is already fixed by* $J_{n-2,2}$ *outside of the total diagonal* $\Delta_n = \{(x_1,\dots,x_n)\in\mathbb{M}^n|x_1=\cdots=x_n\}$. Namely, if not all points coincide, we may separate them into two nonempty sets which are in the past and the future, respectively, of a suitable Cauchy surface. If the last argument x_n is in the past, then $R_{n-1,1}(x_1,\dots,x_n)$ vanishes due to the support properties of retarded products. If, on the other hand, x_n is in the future and x_k for some

$k \neq n$ is in the past, then the retarded product vanishes if the arguments x_k and x_n are permuted, hence in this case we find

$$R_{n-1,1}(x_1, ..., x_{n-1}, x_n) = J_{n-2,2}(x_1, ...\hat{k}..., x_{n-1}, x_k, x_n) \quad \text{if} \quad x_k \notin x_n + \bar{V}_+ \ . \tag{3.1}$$

Moreover, it is only the totally symmetric part S_n of $R_{n-1,1}$ which is not completely fixed by lower orders,

$$S_n(x_1, \ldots, x_n) \stackrel{\text{def}}{=} \frac{1}{n} \sum_{k=1}^{n} R_{n-1,1}(x_{k+1}, \ldots, x_n, x_1, \ldots, x_k) \ . \tag{3.2}$$

This follows again from the GLZ relation which yields

$$R_{n-1,1}(x_1, \ldots, x_n) = S_n(x_1, \ldots, x_n) + \frac{1}{n} \sum_{k=1}^{n-1} J_{n-2,2}(x_{k+1}, \ldots, x_{n-1}, x_1, \ldots, x_k, x_n) \ .$$

We now want to prove the **existence** of $R_{n-1,1}$. In this Subsect. we give the first step: we check that the above findings define a distribution on the complement of the total diagonal which we denote by[11] $R^\circ_{n-1,1}$. For this purpose, in view of (3.1) and the sheaf theorem on distributions, it is sufficient to check that in case two of the first $n-1$ arguments, say x, y, are different from the last argument z and both in the past of z, then

$$J_{n-2,2}(\ldots, x, y, z) = J_{n-2,2}(\ldots, y, x, z) \ .$$

But by the Jacobi identity (2.29), the difference is equal to

$$J_{n-2,2}(\ldots, z, x, y)$$

which vanishes since z is neither in the past of x nor y. So, $R^\circ_{n-1,1}$ is well defined by (3.1) and fulfills the axioms Symmetry and Causality by construction.

In the next step we check that $R^\circ_{n-1,1}$ satisfies all other axioms. The only nontrivial points are the GLZ relation and the Scaling. Concerning the former we have to prove that

$$R^\circ_{n-1,1}(\ldots, x, y, z) - R^\circ_{n-1,1}(\ldots, x, z, y) = J_{n-2,2}(\ldots, x, y, z)$$

[11] The construction of $R^\circ_{n-1,1}$ (from the retarded products of lower orders) can be done for $R^{\circ(m)}_{n-1,1}$ as well as for $R^{\circ(m,\mu)}_{n-1,1}$, the results are related by (2.37).

holds whenever the points x, y, z are not identical. If $y \neq z$ then $y \notin z + \overline{V}_+$ or $y \notin z + \overline{V}_-$. In these cases the assertion follows from the construction (3.1) of $R^\circ_{n-1,1}$. So it remains to treat the case $x \neq y \wedge x \neq z$, i.e. when $x \notin y + \overline{V}_\epsilon \cup z + \overline{V}_{\epsilon'}$ with $\epsilon, \epsilon' \in \{+, -\}$. In the case $\epsilon = \epsilon' = -$ all terms in the GLZ relation vanish (by construction of $R^\circ_{n-1,1}$ and due to the support properties of J). In the case $\epsilon = -$ and $\epsilon' = +$ we analogously find $R^\circ(\ldots, x, z, y) = 0$ and $J(\ldots, z, x, y) = 0$. So, by the Jacobi identity the assertion (3.1) becomes $R^\circ(\ldots, x, y, z) = J(\ldots, y, x, z)$ which is the construction (3.1) of $R^\circ_{n-1,1}$. The case $\epsilon = +$ and $\epsilon' = -$ is analogous. Finally, for $\epsilon = \epsilon' = +$ we apply again the Jacobi identity to the right side and find two terms which are just by (3.1) the retarded products on the left side.

We turn to the scaling. We show that $\sigma_\rho \circ J^{(\rho^{-1}m)}_{n-2,2} \circ (\sigma_\rho^{-1})^{\otimes n}$ has polynomial behavior in $\log \rho$. This then implies the same property for $R^{\circ(m)}_{n-1,1}$, since the causal relations are scale invariant. (And from that it follows that also $R^{\circ(m,\mu)}_{n-1,1}$ scales almost homogeneously, analogously to (2.49).) By definition, $J_{n-2,2}$ is a sum of $\star$-products of retarded products $R_{k-1,1}$ and $R_{n-k-1,1}$, $k = 1, \ldots n-1$, which scale by assumption with a polynomial behavior in $\log \rho$. The statement follows now from the fact that σ_ρ is a $\star$-algebra isomorphism from $\mathcal{A}^{(\rho^{-1}m)}$ to $\mathcal{A}^{(m)}$.

3.2 Extension to the total diagonal; the Action Ward Identity

We now come to the main step in renormalization, namely the **extension of the symmetric part S°_n of $R^\circ_{n-1,1}$ (3.2) to a distribution on $\mathbb{M}^n$**. For the construction of $R^\circ_{n-1,1}$ the normalization conditions (i.e. the axioms Action Ward Identity, Covariance, Field Independence, Unitarity, Field equation, Smoothness in $m \geq 0$ and Scaling) have not been needed, but they give guidance how to do the extension of S°_n and reduce the non-uniqueness drastically. In particular the expansion (2.35)-(2.36) (i.e. the axiom Field Independence) and Covariance for translations simplify the problem to the extension of the symmetric part $s^\circ_n(A_1, \ldots)(x_1 - x_n, \ldots)$ of the distribution[12] $r^\circ_{n-1,1}(A_1, \ldots)(x_1 - x_n, \ldots)$ from $\mathcal{D}(\mathbb{R}^{d(n-1)} \setminus \{0\})$ to $\mathcal{D}(\mathbb{R}^{d(n-1)})$. This is the crucial problem of perturbative renormalization, since it is this step which

[12] Using translation invariance of ω_0 we denote $\omega_0(H(A_1(x_1), \ldots A_n(x_n)))$ by $h(A_1, \ldots A_n)(x_1 - x_n, \ldots)$ for $H = R_{n-1,1}$, $R^\circ_{n-1,1}$, S_n, S°_n etc. .

is non-unique and which is the source of anomalies. Since the Smoothness axiom applies for the $R^{(m,\mu)}$-products, but not for the $R^{(m)}$-products, the extension is done for $S_n^{\circ(m,\mu)}$. From the resulting $S_n^{(m,\mu)}$, the extension $S_n^{(m)}$ of $S_n^{\circ(m)}$ is obtained by (2.37).

The basic idea to fulfill the **Action Ward Identity** goes as follows: since $\partial^\mu_{x_l} s_n(\ldots, A_l, \ldots)$ is an extension of $\partial^\mu_{x_l} s^\circ_n(\ldots, A_l, \ldots) = s^\circ_n(\ldots, \partial^\mu A_l, \ldots)$ we may define $s_n(\ldots, \partial^\mu A_l, \ldots) \stackrel{\text{def}}{=} \partial^\mu_{x_l} s_n(\ldots, A_l, \ldots)$, provided $s_n(\ldots, A_l, \ldots)$ was already constructed. We are now going to show that this can be done without running into inconsistencies. Namely, the fields in $\mathcal{P}$ are of the form

$$A(x) = \sum_{n=0}^{N} p_n(\partial^1, \ldots, \partial^n)\varphi(x_1)\cdots\varphi(x_n)|_{x_1=\cdots=x_n=x}$$

with polynomials p_n in the derivatives $\partial^k_\mu = \frac{\partial}{\partial x^\mu_k}, k = 1, \ldots, n, \mu = 0, \ldots, d-1$, which are symmetric in the upper indices $k = 1, \ldots, n$. The polynomials p_n are uniquely determined by A.

Now let $\partial_\mu = \sum_{k=1}^n \partial^k_\mu$ denote the derivatives with respect to the center of mass coordinates and $\partial^{ij}_\mu = \partial^i_\mu - \partial^j_\mu, 1 \le i < j \le n$ the relative derivatives. The crucial observation is now that the vector space P_n of all symmetric polynomials p_n is isomorphic to the tensor product of the space P^{com} of polynomials $p(\partial)$ of the center of mass derivatives and the space P^{rel}_n of symmetric polynomials $p_n(\partial^{ij}, 1 \le i < j \le n)$ of the relative derivatives. (Symmetry is meant with respect to permutations $\sigma(\partial^{ij}) := \partial^{\sigma(i)\,\sigma(j)}, \sigma \in S_n$, where $\partial^{ij} \equiv -\partial^{ji}$.) The argument is straightforward for the unsymmetrized polynomials. Namely, the independent variables ∂^{in}, $i = 1, \ldots, n-1$ generate a polynomial algebra $\tilde{P}^{\text{rel}}_n$, and the linear map

$$\alpha : \left\{ \begin{array}{ccc} P^{\text{com}} \otimes \tilde{P}^{\text{rel}}_n & \longrightarrow & \bigvee\{\partial^1, ..., \partial^n\} \\ \alpha(\partial \otimes 1) = \sum_{i=1}^n \partial^i & , & \alpha(1 \otimes \partial^{in}) = \partial^i - \partial^n \end{array} \right. \tag{3.3}$$

is an isomorphism onto the polynomial algebra generated by ∂^i, $i = 1, \ldots, n$.

This isomorphism intertwines the actions of the permutation group which are induced by the permutation of indices (on $\tilde{P}^{\text{rel}}_n$ the action is given by

$$\sigma(\partial^{in}) = \partial^{\sigma(i)\,n} - \partial^{\sigma(n)\,n}$$

with $\partial^{nn} = 0$) and thus restricts to an isomorphism of the invariant subspaces. Interpreting P^{rel}_n as a subspace of $\tilde{P}^{\text{rel}}_n$ (by the obvious identification $\partial^{ij} \equiv \partial^{in} - \partial^{jn}$), the invariant subspace of $P^{\text{com}} \otimes \tilde{P}^{\text{rel}}_n$ is just $P^{\text{com}} \otimes P^{\text{rel}}_n$.

We therefore use the space of balanced derivatives of fields [10]

$$\mathcal{P}_{\mathrm{bal}} \stackrel{\mathrm{def}}{=} \{p_n(\partial^{ij}, 1 \leq i < j \leq n)\varphi(x_1)\cdots\varphi(x_n)|_{x_1=\cdots=x_n=x} \,|\, p_n \in P_n^{\mathrm{rel}} \ , \ n \in \mathbb{N}\} \ . \tag{3.4}$$

and obtain the isomorphism of vector spaces

$$\mathcal{P} \cong P^{\mathrm{com}}(\partial) \otimes \mathcal{P}_{\mathrm{bal}} \ .$$

In other words, every $A \in \mathcal{P}$ can *uniquely* be written as

$$A = \sum_{i=1}^{N} p_i(\partial)\, B_i \ , \quad p_i(\partial) \in P^{\mathrm{com}}(\partial) \ , \quad B_i \in \mathcal{P}_{\mathrm{bal}} \ , \quad N < \infty \ .$$

Applying this result to (2.8) we obtain

Proposition 2. *Let F be a local functional. Then there exists a unique $h \in \mathcal{D}(\mathbb{M}, \mathcal{P}_{bal})$ with $F = \int dx\, h(x)$.*

The Action Ward Identity can now simply be fulfilled by performing the extension first only for balanced fields $B \in \mathcal{P}_{\mathrm{bal}}$ and by using the AWI and linearity for the definition of the extension for general fields $A \in \mathcal{P}$. By construction this yields, in every entry, a linear map from $P^{\mathrm{com}}(\partial) \otimes \mathcal{P}_{\mathrm{bal}} \cong \mathcal{P}$ to distributions on $\mathbb{M}$.

Next we recall from the literature the main statements about the extension of a distribution $t^\circ \in \mathcal{D}'(\mathbb{R}^k \setminus \{0\})$ to $t \in \mathcal{D}'(\mathbb{R}^k)$ and give some completions. The existence and uniqueness of t can be answered in terms of Steinmann's *scaling degree* [42] of t°. The latter is defined by

$$\mathrm{sd}(f) \stackrel{\mathrm{def}}{=} \inf\{\delta \in \mathbb{R} | \lim_{\rho \downarrow 0} \rho^\delta f(\rho x) = 0\}, \quad f \in \mathcal{D}'(\mathbb{R}^k) \mathrm{or} f \in \mathcal{D}'(\mathbb{R}^k \setminus \{0\}). \tag{3.5}$$

We immediately see that a distribution which scales almost homogeneously with degree D and an arbitrary power $N < \infty$ has scaling degree D. In addition one easily verifies

$$\mathrm{sd}(\partial^a f) \leq \mathrm{sd}(f) + |a|, \quad \mathrm{sd}(x^b f) \leq \mathrm{sd}(f) - |b| \quad \text{and} \quad \mathrm{sd}(\partial^a \delta^{(k)}) = k + |a|, \tag{3.6}$$

where $a, b \in \mathbb{N}_0^k$, $|a| \equiv \sum_{j=1}^k a_j$. Returning to the extension of t°, the definition (3.5) implies immediately: $\mathrm{sd}(t) \geq \mathrm{sd}(t^\circ)$. We are looking for extensions which do not increase the scaling degree.[13]

Theorem 3. *[7]: Let $t^\circ \in \mathcal{D}'(\mathbb{R}^k \setminus \{0\})$.*
(a) If $\mathrm{sd}(t^\circ) < k$, there exists a unique extension $t \in \mathcal{D}'(\mathbb{R}^k)$ with $\mathrm{sd}(t) = \mathrm{sd}(t^\circ)$.
(b) If $k \leq \mathrm{sd}(t^\circ) < \infty$ there exist several extensions $t \in \mathcal{D}'(\mathbb{R}^k)$ with $\mathrm{sd}(t) = \mathrm{sd}(t^\circ)$. Given a particular solution t_0, the most general solution reads

$$t = t_0 + \sum_{|a| \leq \mathrm{sd}(t^\circ) - k} C_a \partial^a \delta^{(k)} \tag{3.8}$$

with arbitrary constants $C_a \in \mathbb{C}$.
(c) If $\mathrm{sd}(t^\circ) = \infty$ there exists no extension $t \in \mathcal{D}'(\mathbb{R}^k)$.

Case (c) is mentioned for mathematical completeness only. It does not appear in our construction, because $r^\circ_{n+1,1}$ scales almost homogeneously with finite degree. However, there are distributions t° with $\mathrm{sd}(t^\circ) = \infty$, e.g. $t^\circ(x) = e^{\frac{1}{|x|}}$. The proof of (a)-(b) given in [7] is based on [18] and it is constructive ('W-extension'). It is sketched in Appendix B.

The W-extension has the disadvantage that in general it does not maintain $\mathcal{L}_+^\uparrow$-covariance for $\mathrm{sd}(t^\circ) > k$. However, by a finite renormalization (which does not increase the scaling degree) Lorentz covariance can be restored (see [18, 42, 44, 40, 4] and also Appendix D).

The W-extension breaks in general also the property of almost homogeneous scaling. However, Hollands and Wald have given a (finite) renormalization prescription to restore also this symmetry, in detail:[14]

[13]In a large part of the literature (e.g. [3, 18, 40]) our axioms 'Smoothness in $m \geq 0$' and 'Scaling' are replaced by the weaker requirement

$$\mathrm{sd}\Big(r_{n-1,1}(A_1, \ldots, A_n)(x_1 - x_n, \ldots)\Big) \leq \sum_{j=1}^n \dim(A_j) \ , \quad \forall A_j \in \mathcal{P}_{\mathrm{hom}} \ , \tag{3.7}$$

or an analogous normalization condition. For the extension problem this amounts (nearly always) to the requirement $\mathrm{sd}(t) = \mathrm{sd}(t^\circ)$. We point out that (3.7) is a condition on the behavior under rescalings of the coordinates in the UV-region only. In contrast, our Scaling axiom is with respect to simultaneous rescalings of the coordinates and the mass, and it must hold for all x and for all $m \geq 0$.

[14]This is a version of Lemma 4.1 in the second paper of [31], which follows from the proof given in that paper. The 'improved Epstein-Glaser renormalization' of [24] maintains the almost homogeneous scaling directly.

Proposition 4. *Let* $t^\circ \in \mathcal{D}'(\mathbb{R}^k \setminus \{0\})$ *scale almost homogeneously with degree* $D \in \mathbb{R}$ *and power* $N \in \mathbb{N}$ *under coordinate rescalings (2.38). Then* t° *has an extension* t *to a distribution on* $\mathbb{R}^k$ *which also scales almost homogeneously with degree* D *and with power* N, *if* $D \notin \mathbb{N}_0 + k$, *and with power* $(N+1)$ *or* N, *if* $D \in \mathbb{N}_0 + k$. *The extension is unique if* $D \notin \mathbb{N}_0 + k$, *otherwise, the most general solution reads: (particular solution)* $+ \sum_{|a|=D-k} C_a \partial^a \delta^{(k)}$, *where the* C_a*'s are arbitrary constants.*

Now we assume that $t^\circ \equiv t^{(m)\circ} \in \mathcal{D}'(\mathbb{R}^{dr} \setminus \{0\})$ is smooth in $m \geq 0$ and scales almost homogeneously with degree D and power N under simultaneous rescalings of the coordinates and the mass m:

$$(\rho\partial_\rho)^{N+1}\Big(\rho^D t^{(\rho^{-1}m)\circ}(\rho y)\Big) = 0 \ . \tag{3.9}$$

- For $m = 0$ the scaling of m is trivial and the Proposition can directly be applied.

- For $m > 0$ the Smoothness in $m \geq 0$ of $t^\circ \equiv t^{(m)\circ}$ ensures the existence of the Taylor expansion[15]

  $$t^{(m)\circ}(y) = \sum_{l=0}^{D-dr} \frac{m^l}{l!} u_l^\circ(y) + m^{[D]-dr+1} t_{\rm red}^{(m)\circ}(y) \tag{3.10}$$

  (where $[D]$ is the integer part of D) with

  $$u_l^\circ(y) \stackrel{\rm def}{=} \frac{\partial^l t^{(m)\circ}(y)}{\partial m^l}|_{m=0} \ , \tag{3.11}$$

  where the 'reduced part' $t_{\rm red}^{(m)\circ}$ is smooth in $m \geq 0$. The almost homogeneous scaling of $t^{(m)\circ}$ (3.9) and the definition of u_l° imply

  $$(\rho\partial_\rho)^{N+1}(\rho^{D-l} u_l^\circ(\rho y)) = 0 \ . \tag{3.12}$$

  Hence, by the Proposition, u_l° has an extension u_l with $(\rho\partial_\rho)^{N+2}(\rho^{D-l}u_l(\rho y)) = 0$.

[15] This is the 'scaling expansion' of Hollands and Wald (given in the second paper of [31]) in the particular simple case of Minkowski space.

For the reduced part we find that $m^{[D]-dr+1}t_{\mathrm{red}}^{(m)\circ}$ scales almost homogeneously with degree D, because all other terms in (3.10) have this property. This gives

$$\rho^{D-[D]+dr-1}t_{\mathrm{red}}^{(m)\circ}(\rho y)=t_{\mathrm{red}}^{(\rho m)\circ}(y)+\sum_{j=1}^{N}(\log\rho)^j l_j^{(\rho m)}(y) \qquad (3.13)$$

for some $l_j^{(m)}\in\mathcal{D}'(\mathbb{R}^{dr}\setminus\{0\})$ which are smooth in $m\geq 0$. Since also $t_{\mathrm{red}}^{(m)\circ}$ is smooth in $m\geq 0$, we conclude

$$\lim_{\rho\to 0}\rho^{dr}t_{\mathrm{red}}^{(m)\circ}(\rho y)=0\ ,\quad \text{i.e.}\quad \mathrm{sd}(t_{\mathrm{red}}^{(m)\circ})<dr\ . \qquad (3.14)$$

With that and with part (a) of Theorem 3 the reduced part $t_{\mathrm{red}}^{(m)\circ}$ can be uniquely extended. Due to the latter, the resulting $t_{\mathrm{red}}^{(m)}$ is also smooth in $m\geq 0$ and also the scaling property (3.13) is maintained. Namely, $(\rho\partial_\rho)^{N+1}\big(\rho^{D-[D]+dr-1}t_{\mathrm{red}}^{(\rho^{-1}m)}(\rho y)\big)$ has support in $\{0\}$ and its scaling degree is less than dr for each fixed $\rho>0$.

Putting together the extensions of the individual terms we get

$$t^{(m)}(y)\stackrel{\mathrm{def}}{=}\sum_{l=0}^{D-dr}\frac{m^l}{l!}u_l(y)+m^{[D]-dr+1}t_{\mathrm{red}}^{(m)}(y)\ . \qquad (3.15)$$

By construction this is an extension of $t^{(m)\circ}$ with the wanted smoothness and scaling properties. It is unique for $D\notin\mathbb{N}_0+dr$. For $D\in\mathbb{N}_0+dr$ the most general solution is obtained by adding

$$\sum_{l=0}^{D-dr}\ \sum_{|a|=D-dr-l}m^l C_{l,a}\partial^a\delta^{(dr)} \qquad (3.16)$$

to a particular solution, with arbitrary constants $C_{l,a}$. Note that the undetermined polynomial (3.16) scales even homogeneously (with degree D). If we would require almost homogeneous scaling only (and not smoothness in $m\geq 0$), terms with $l<0$ would be admitted in (3.16). An extension with such terms increases the scaling degree: $\mathrm{sd}(t)>\mathrm{sd}(t^\circ)$, cf. footnote 13.

If $t^{(m)\circ}$ (3.9) is additionally Lorentz invariant, a slight modification of the usual proofs of Lorentz invariance [18, 42, 44, 40, 4] yields that $t^{(m)}$ (3.15) can be chosen to be also Lorentz invariant (Appendix D). The conditions Unitarity and Symmetry can easily be included, too (see e.g. [18]).

With this general knowledge about the extension of a distribution to a point we return to the extension of $s_n^{(m,\mu)\circ}$. For $A_1, \dots, A_n \in \mathcal{P}_{\rm bal} \cap \mathcal{P}_{\rm hom}$ the distribution $s_n^{(m,\mu)\circ}(A_1, \dots, A_n)$ fulfills (3.9) with degree $D = \sum_j \dim(A_j)$ and some power $N < \infty$, and we can proceed as follows:

Step 1: We first extend the distributions $s_n^{(m,\mu)\circ}$ for homogeneous (2.42) balanced fields only, by applying the above given procedure, including the finite renormalizations which restore Lorentz covariance, almost homogeneous scaling (3.9), Symmetry, Unitarity and maintain Smoothness in $m \geq 0$. Furthermore, the global inner symmetries (in our case the field parity (2.10)) can be preserved, cf. Appendix D.

Step 2: From that we construct $s_n^{(m,\mu)}$ for all fields by using linearity and the Action Ward Identity.

Step 3: Finally we construct $S_n^{(m,\mu)}$ by means of the Taylor expansion (2.35)-(2.36),

$$S_n^{(m,\mu)}(A_1(x_1), \dots, A_n(x_n)) \stackrel{\rm def}{=} \sum_{l_1 \dots} \frac{1}{l_1! \dots} \int dx_{11} \dots dx_{1l_1} \dots$$

$$s_n^{(m,\mu)}\Big(\frac{\delta^{l_1} A_1(x_1)}{\delta\varphi(x_{11}) \dots \delta\varphi(x_{1l_1})}, \dots\Big)\varphi(x_{11}) \dots \varphi(x_{1l_1}) \dots \ . \tag{3.17}$$

By construction $S_n^{(m,\mu)}$ is linear in the fields and fulfills the axioms Covariance with respect to translations and Field Independence. The properties of $s_n^{(m,\mu)}$ established in the steps 1 and 2 imply that $S_n^{(m,\mu)}$ fulfills the corresponding axioms. In particular, by using $\partial_y \frac{\delta A(y)}{\delta\varphi(x)} = \frac{\delta \partial A(y)}{\delta\varphi(x)}$, we see from (3.17) that $S_n^{(m,\mu)}$ satisfies the AWI.

From a *particular* solution, the *general* solution is obtained by adding an arbitrary local polynomial of the form (3.16) to $s_n^{(m,\mu)}(A_1, \dots, A_n)$ which respects also linearity in the fields, Symmetry, Unitarity and the AWI. $R_{n-1,1}^{(m)}$ is obtained from $R_{n-1,1}^{(m,\mu)}$ by (2.37).

The construction given so far yields the most general solution $R^{(m,\mu)}$ and $R^{(m)}$ of the axioms of Sect. 2 except the Field equation (2.12). (Since the latter has precisely the same form for both kind of retarded products the following procedure applies to both kinds and the results are related by (2.37).) Due to the expansion (2.35)-(2.36) the Field equation is *equivalent* to

$$r_{n-1,1}(F_1,...,F_{n-1},\varphi(h)) =$$
$$-\int dx\, h(x)\int dy\,\Delta^{\rm ret}(x-y)\sum_{k=1}^{n-1} r_{n-2,1}(F_1,...\hat{k}...,F_{n-1},\frac{\delta F_k}{\delta\varphi(y)})\ , \tag{3.18}$$

for all $n\geq 2$, $F_1,\ldots,F_{n-1}\in\mathcal{F}_{\rm loc}$, and $h\in\mathcal{D}(\mathbb{M})$. The right side gives an extension of $r^\circ_{n-1,1}(F_1,...,F_{n-1},\varphi(h))$, because the Field equation holds outside the total diagonal. It is Lorentz covariant, symmetric in the first $(n-1)$ factors, unitary, smooth in $m\geq 0$, scales almost homogeneously (even with power $\leq (n-2)$) and respects the AWI. From (3.18) and the inductively known $\omega_0(J_{n-2,2}(F_1,\ldots,F_{n-1},\varphi(h)))$ we obtain $r_{n-1,1}(F_1,\ldots,\varphi(h),\ldots,F_{n-1})$ by using the GLZ relation and the Symmetry in the first $(n-1)$ factors.[16] By construction this yields an extension of $r^\circ_{n-1,1}(F_1,\ldots,\varphi(h),\ldots,F_{n-1})$ which also satisfies all axioms. With that $s_n(\ldots,\varphi(h))$ (3.2) is uniquely determined in terms of the inductively known $r_{n-2,1}$.

So, in order to fulfill the Field equation we modify the step 1 as follows: $s_n(A_1,\ldots,A_{n-1},\varphi)$, $A_1,\ldots,A_{n-1}\in\mathcal{P}_{\rm bal}$, is uniquely given by the Field equation in the just described way and fulfills the required properties. However, the construction of $s_n(A_1,\ldots,A_n)$ remains unchanged if $A_1,\ldots,A_n$ are all of at least second order in φ and its partial derivatives. (If at least one factor is a C-number the retarded product vanishes and hence also s_n, see part (C) of Lemma 1 in [12].) Finally steps 2 and 3 are done as before.

Summing up we have proved:

Theorem 5. *There exist retarded products which fulfill all axioms of Sect. 2.*

Example: **setting-sun diagram $r^{(m,\mu)}_{1,1}(\varphi^3,\varphi^3)$ for $d=4$:** the explicit calculation of a diagram requires usually somewhat less work if the extension is done directly for r° (and not for its symmetric part s°). By using the GLZ relation and Causality we obtain

$$r^{\circ(m,\mu)}(y)\equiv r^{\circ(m,\mu)}_{1,1}(\varphi^3,\varphi^3)(y)=-6i\left(H^\mu_m(y)^3-H^\mu_m(-y)^3\right)\Theta(-y^0)\ . \tag{3.19}$$

[16]The result is given in part (B) of Lemma 1 in [12].

From (A.8) we can read off the first terms of the Taylor expansion in m^2 of $H^\mu_m(y)$:

$$H^\mu_m(y) = D^+(y) + m^2 \left[\log(-\mu^2(y^2 - iy^0 0))\, f(0) + F(0)\right] + h_m(y) \ , \quad (3.20)$$

where $D^+(y) \equiv -(4\pi^2(y^2 - iy^0 0))^{-1}$ is the massless two-point function and $h_m(y)$ is of order $\mathcal{O}(m^4)$ and has scaling degree $\mathrm{sd}(h_m) < 0$. With that we get the scaling expansion (3.10) of (3.19):

$$r^{\circ(m,\mu)}(y) = u^\circ_0(y) + \frac{m^2}{2} u^{\circ(\mu)}_2(y) + r^{\circ(m,\mu)}_{\mathrm{red}}(y) \ , \quad (3.21)$$

where

$$u^\circ_0(y) = -6i \left(D^+(y)^3 - D^+(-y)^3\right) \Theta(-y^0) \ ,$$
$$u^{\circ(\mu)}_2(y) = -36i \left(D^+(y)^2 \left[\log(-\mu^2(y^2 - iy^0 0))\, f(0) + F(0)\right] - (y \to -y)\right) \Theta(-y^0) \ . \quad (3.22)$$

The power of the almost homogeneous scaling (3.9) (with degree 6) is the power of $\log(\mu^2 ...)$. It is different for the individual terms: it is 0 for u°_0, 1 for u°_2 and 3 for $r^{\circ(m,\mu)}_{\mathrm{red}}$ respectively. In contrast to the reduced part $r^{\circ(m,\mu)}_{\mathrm{red}}$, the renormalization of u°_0 and $u^{\circ(\mu)}_2$ is non-trivial and it increases the power of the almost homogeneous scaling by 1. The extension of these two terms is given in Appendix B by using *differential renormalization*. An alternative method, which relies on the Källen-Lehmann representation, is applied to the massless fish and setting-sun diagram in Appendix C.

We now focus on the power N of the almost homogeneous scaling (3.9). The preceding example shows that, in the scaling expansion of $r^{\circ(m,\mu)}$ (or $s^{\circ(m,\mu)}$), the terms for which N may be increased in the extension, are not the terms with the maximal value of N. The proof of part (ii) of the following Proposition is based on this observation.

Proposition 6. *(i) If the number d of space time dimensions is* **odd***, the power N of the almost homogeneous scaling of $R^{(m)}_{n-1,1} \equiv R^{(m,\mu)}_{n-1,1}$ is smaller than n, i.e. $R^{(m)}_{n-1,1\,\rho}$ (2.48) is a polynomial of* $(\log \rho)$ *with degree less than n.*

(ii) For $d = 4$ the power N of the almost homogeneous scaling (3.9) of

$$r^{(m,\mu)}_{n-1,1}(A_1, \ldots, A_n) \ , \quad \textit{with} \quad A_j = \prod_{s=1}^{l_j} \partial^{a_{js}} \varphi \tag{3.23}$$

and

$$\sum_{j=1}^{n} \dim(A_j) \leq 4(n-1) + 3 \ , \tag{3.24}$$

is bounded by

$$N \leq \frac{1}{2} \sum_{j=1}^{n} l_j \qquad \Big(\leq \frac{1}{2} \sum_{j=1}^{n} \dim(A_j)\Big) \ . \tag{3.25}$$

Note that $\frac{1}{2} \sum_{j=1}^{n} l_j$ is the number of (internal) lines in the corresponding Feynman diagram. Due to the expansion (2.35)-(2.36) and the fact that $\prod_j \partial^{a_j} \varphi(x_{i_j})$ scales homogeneously, part (ii) implies that the power N of the almost homogeneous scaling of $R^{(m,\mu)}_{n-1,1}(A_1, \ldots, A_n)$ (with A_j of the mentioned kind) is also bounded by (3.25). The restriction (3.24) on the A_j's is e.g. satisfied for interacting fields $A_{\mathcal{L}(g)}$ if $\dim(A) \leq 3$ and $\mathcal{L}$ is renormalizable by power counting, cf. Sect. 4.1.

Proof. Part (i): $R_{0,1}(A) = A$ scales homogeneously (2.44). Following our inductive construction, one verifies that an increase of N may happen in the extension $s^\circ_n \to s_n$ only. Hence, by Proposition 4, N is increased at most by 1 in each inductive step.

Part (ii): We give the proof for $A_j = \varphi^{l_j}$ only. The generalization to field polynomials with derivatives is straightforward, it gives only notational complications. In our inductive construction of the $R^{(m,\mu)}$-products N may now be increased also in the construction of $j^{(m,\mu)}_{n-2,2}$, since the GLZ relation uses the modified star product $\star_{m,\mu}$. The vacuum expectation value of the GLZ relation reads

$$\omega_0\Big(\Big\{R^{(m,\mu)}\big(\varphi^{l_1}(x_1), ..., \varphi^{l_k}(x_k)\big), R^{(m,\mu)}\big(\varphi^{l_{k+1}}(y_1), ..., \varphi^{l_n}(y_{n-k})\big)\Big\}_{\star_{m,\mu}}\Big) =$$
$$\sum ... r^{(m,\mu)}\big(\varphi^{l_1-p_1}(x_1), ..., \varphi^{l_k-p_k}(x_k)\big)\, r^{(m,\mu)}\big(\varphi^{l_{k+1}-p_{k+1}}(y_1), ..., \varphi^{l_n-p_n}(y_{n-k})\big)$$
$$\cdot \Big(\prod_{j=1}^{p} H^\mu_m(x_{i_j} - y_{i_j}) - \prod_{j=1}^{p} H^\mu_m(y_{i_j} - x_{i_j})\Big) \ , \tag{3.26}$$

where $p \equiv \frac{1}{2}(p_1 + ... + p_n)$. By using the inductive assumption and the fact that H^μ_m scales almost homogeneously with power 1 (A.12) [17] we find that for each term (on the right side) N is bounded by

$$\frac{1}{2}\sum_{j=1}^{k}(l_j - p_j) + \frac{1}{2}\sum_{j=k+1}^{n}(l_j - p_j) + p = \frac{1}{2}\sum_{j=1}^{n} l_j \ . \tag{3.27}$$

We turn to the extension $s^\circ_n \to s_n$. The scaling degree of each term in (3.26) is bounded by

$$\mathrm{sd}(\ldots) \le \sum_{j=1}^{n} \dim(A_j) \le 4(n-1) + 3 \ . \tag{3.28}$$

- If $p = 1$ the extension is trivial and, hence, the power N is not increased in this step.
- If $p \ge 2$ and the scaling degree is ≥ 0, the extension may increase N by 1. The terms on the right side of (3.26) with the *maximal* value of N have the property that $-m^2\, f(m^2y^2)\log(m^2/\mu^2)$ is substituted for $H^\mu_m(y)$ (A.11) in *all* H^μ_m's. Therefore, the scaling degree of these terms is lowered by $2p \ge 4$, i.e. it is $< 4(n-1)$, and we are in the case of trivial extension. We conclude that in the extension $s^{\circ(m,\mu)}_n(\varphi^{l_1}, ...) \to s^{(m,\mu)}_n(\varphi^{l_1}, ...)$ the corresponding value of N is not increased, i.e. N is still bounded by (3.27).

□

In $d = 4 + 2k$ ($k \in \mathbb{N}$) space time dimensions analogous bounds on the power N of the almost homogeneous scaling of the $R^{(m,\mu)}$-products can be derived by the same method.

4 Non-uniqueness

4.1 Counting the indeterminate parameters before the adiabatic limit

In contrast to the literature we count the indeterminate parameters *without performing the adiabatic limit.*

[17] An essential ingredient of the generalization to field polynomials with derivatives is that also partial derivatives $\partial^a H^\mu_m$ ($a \in \mathbb{N}^4_0$) scale almost homogeneously with power 1.

The interacting fields $A_{\mathcal{L}(g)}(x) = \sum_{n=0}^{\infty} \frac{1}{n!} R_{n,1}((\mathcal{L}(g))^{\otimes n}; A(x))$ are left with an indefiniteness coming from the extension of the symmetric part of $r^{\circ}_{n,1}$ to the origin (in relative coordinates). In general the normalization conditions restrict this indefiniteness only, they to do not remove it completely.

Let $\mathcal{L} \in \mathcal{P}_{\rm bal}$ and let $N(\mathcal{L}, A, n)$ be the number of indeterminate parameters (i.e. the constants C_a in (3.8) or (B.6)) in $R_{n,1}(\mathcal{L}(g))^{\otimes n}; A(x))$ coming from the inductive step $(n-1,1) \to (n,1)$. This number depends on the choice of the normalization conditions. In the following we presume the axioms given in Sect. 2 except Lorentz covariance, Unitarity and Field equation. We will prove

Proposition 7. *Let $\mathcal{L} \in \mathcal{P}_{bal}$.*

(a) $N(\mathcal{L}, A, n)$ is bounded in n $\forall A \in \mathcal{P}$ fixed, iff $\dim(\mathcal{L}) \leq d$.

(b) For all $A \in \mathcal{P}$ there exists $n(A)$ such that $N(\mathcal{L}, A, n) = 0 \;\; \forall n > n(A)$, iff $\dim(\mathcal{L}) < d$.

An interaction $\mathcal{L}$ with the property *(a)* of $n \mapsto N(\mathcal{L}, A, n)$ is called 'renormalizable by power counting'. In the literature (also in causal perturbation theory [18, 40, 41]) the counting of indeterminate parameters is done in terms of the S-matrix *in the adiabatic limit* (i.e. $\sum_n \frac{1}{n!} \lim_{g\to 1} T_n((\mathcal{L}(g))^{\otimes n})$), and the corresponding version of the Proposition can be proved rather easily, see e.g. Sect. 28.1 of [3]. It does not make an essential difference that we count in terms of retarded products. But, since we do not perform the adiabatic limit, our discussion is more involved.

Proof. Due to

$$\frac{\delta A(x)}{\delta \varphi(y)} = \sum_a \frac{\partial A}{\partial(\partial^a \varphi)}(x)(-1)^{|a|}\partial^a \delta(y-x) \tag{4.1}$$

we understand by the sub-polynomials U of $A \in \mathcal{P}$ all non-vanishing polynomials $U \equiv \frac{\partial^k A}{\partial(\partial^{a_1}\varphi)\dots\partial(\partial^{a_k}\varphi)}$, $k \in \mathbb{N}_0$, $a_j \in \mathbb{N}_0^d$ and we write $U \subset A$. Ignoring the AWI, the indefiniteness of $R_{n,1}((\mathcal{L}(g))^{\otimes n}; A(x))$ is precisely the indefiniteness of all C-number distributions $r_{n,1}(U_1, ..., U_n; U)$, $U_1, ..., U_n \subset \mathcal{L}$ and $U \subset A$, due to the expansion (2.35)-(2.36). Note

$$0 \leq \dim(U_j) \leq \dim(\mathcal{L}), \qquad \text{and} \qquad 0 \leq \dim(U) \leq \dim(A). \tag{4.2}$$

The indefiniteness of $r_{n,1}(U_1,...,U_n;U)(x_1-x,...,x_n-x)$ is a polynomial (3.16)

$$\sum_{|a|\leq\omega(U_1,...,U_n;U)} C^n_a(U_1,...,U_n;U)\partial^a\delta(x_1-x,...,x_n-x) \tag{4.3}$$

which is invariant under permutations of $(U_1,x_1),...,(U_n,x_n),(U,x)$, where

$$\begin{aligned}\omega(U_1,...,U_n;U) &= \sum_{j=1}^n \dim(U_j)+\dim(U)-dn \\ &\leq \dim(U)+n(\dim(\mathcal{L})-d)\ .\end{aligned} \tag{4.4}$$

For $m=0$ the sum (4.3) runs over $|a|=\omega$ only; this simplifies the proof.

With (4.2)-(4.4) the only non-obvious statement of the Proposition is that for an interaction $\mathcal{L}$ with $\dim(\mathcal{L})=d$ the function $n\mapsto N(\mathcal{L},A,n)$ is bounded $\forall A\in\mathcal{P}$. This statement holds even true if the AWI is not required, and we are now going to prove it under this weaker supposition. The boundedness (in n) of $\omega(U_1,...,U_n;U)$ *alone*, i.e.

$$\omega(U_1,...,U_n;U)=\dim(U)-\sum_{j=1}^n(d-\dim(U_j))\leq\dim(U)\leq\dim(A)\quad\forall n\in\mathbb{N} \tag{4.5}$$

and $\forall U_j\subset\mathcal{L}$, $U\subset A$, does not imply the assertion, because

(i) the number of terms $r_{n,1}(U_1,...,U_n;U)$, $U_j\subset\mathcal{L}$, $U\subset A$, is increasing with n;

(ii) the number of indices $a\in\mathbb{N}_0^{dn}$ with $|a|\leq\omega_0$, ω_0 fixed, see (4.3), is also increasing with n. (Hence, one might expect that there is e.g. an increasing number of constants $C^n_a(\mathcal{L},...,\mathcal{L},A)$.)

Now let $A\in\mathcal{P}$ be fixed. (i) is no problem, since there are indeterminate parameters in the $r_{n,1}(U_1,...,U_n;U)$ for $\omega(U_1,...,U_n;U)\geq 0$ (4.5) only. Hence, we solely need to consider

$$\mathcal{R}_n\stackrel{\text{def}}{=}\{r_{n,1}(U_1,...,U_l,\mathcal{L},...,\mathcal{L};U)\,|\,U_1,...,U_l\subset\mathcal{L},\ U\subset A\}\ , \tag{4.6}$$

where l is given by $l\cdot\dim(\varphi)=\dim(A)$. However, the number of elements of $\mathcal{R}_n$ is constant for all $n\geq l$.

To invalidate the objection (ii) let $U_1,...,U_l$ and U be fixed. Because

$$r_{n,1}(U_1,...,U_l,\mathcal{L},...,\mathcal{L};U)(y_1,...,y_n),\quad y_j\stackrel{\text{def}}{=}x_j-x, \tag{4.7}$$

is *symmetrical* in $y_{l+1},...,y_n$, the number of constants

$$C_a^n(U_1,...,U_l,\mathcal{L},...,\mathcal{L};U) \quad \text{with} \quad a\in\mathbb{N}_0^{dn},\ |a|\leq \dim(U)-\sum_{j=1}^{l}(d-\dim(U_j)) \tag{4.8}$$

is bounded in n. We use here a modified version of the fact that the number of coefficients in the *symmetrical* polynomials $P(z_1,...,z_m)$ ($z_j\in\mathbb{R}$), $m\in\mathbb{N}$, of a fixed degree becomes independent of the number m of variables z_j, if m is big enough.[18]

Summing up we find that $N(\mathcal{L},A,n)$ is bounded in n for any fixed $A\in\mathcal{P}$. □

4.2 Main Theorem of Perturbative Renormalization

It is one of the main insights of renormalization theory that the ambiguities of the renormalization process can be absorbed in a redefinition of the parameters of the given model. In causal perturbation theory this was termed **Main Theorem of Renormalization** [44]. Different versions of this theorem may be found in [46, 22, 3, 44, 38, 25]. But there, in contrast to the formulation of renormalization in terms of the action functional, the parameters of a model are test functions. Therefore, the renormalization group which governs the change of parameters is more complicated, and it is only in the adiabatic limit that the more standard version of the renormalization group will be recovered.

Fortunately, the algebraic adiabatic limit [7] is sufficient for this purpose, as was first shown by Hollands and Wald [32]. In this way, one finds an intrinsically local construction of the renormalization group which is suited for theories on curved spacetime and for theories with a bad infrared behavior.

Here we give a slightly streamlined proof of the Main Theorem in the framework of retarded products. In Sect. 5 we discuss the consequences for

[18] This becomes obvious by listing the symmetrical polynomials in $z_1,...,z_m$ which are homogeneous of degree k for the lowest values of k: e.g. for $k=4$ and for all $m\geq 4$ a basis is given by

$$\begin{aligned} P_1=C_1\mathcal{S}z_1^4\ ,\quad P_2=C_2\mathcal{S}z_1^3z_2\ ,P_3=C_3\mathcal{S}z_1^2z_2^2\ ,\\ P_4=C_4\mathcal{S}z_1^2z_2z_3\ ,\quad P_5=C_5\mathcal{S}z_1z_2z_3z_4\ , \end{aligned} \tag{4.9}$$

where $\mathcal{S}f(z_1,\ldots,z_m)\equiv\frac{1}{m!}\sum_{\pi\in S_m}f(z_{\pi 1},\ldots,z_{\pi m})$.

the algebraic adiabatic limit.

Theorem 8. *(i) Let $\hat{R}$, $R : T\mathcal{F}_{\rm loc} \longrightarrow \mathcal{F}$ be linear maps which satisfy the axioms Symmetry, Initial Condition, Causality and GLZ for retarded products. Then there exists a unique symmetric linear map*

$$D : T\mathcal{F}_{\rm loc} \longrightarrow \mathcal{F}_{\rm loc} \tag{4.10}$$

with $D(1) = 0$ such that for all $F, S \in \mathcal{F}_{\rm loc}$ the following intertwining relation holds (in the sense of formal power series in λ)

$$\hat{R}(e_\otimes^{\lambda S}, F) = R(e_\otimes^{D(e_\otimes^{\lambda S})}, D(e_\otimes^{\lambda S} \otimes F)) \ . \tag{4.11}$$

Moreover, D satisfies the conditions

(a) $D(F) = F$, $F \in \mathcal{F}_{loc}$.

(b)

$$\mathrm{supp}\ \frac{\delta\, D(F_1 \otimes \ldots \otimes F_n)}{\delta\varphi} \subset \bigcap_i \mathrm{supp}\ \frac{\delta F_i}{\delta\varphi}, \quad F_i \in \mathcal{F}_{loc}\ . \tag{4.12}$$

(ii) If in addition, R and $\hat{R}$ satisfy some of the conditions Covariance, Unitarity, Field Independence and Field Equation, then D has the corresponding properties, i.e. it is covariant, hermitian [19], has no explicit dependence on φ,

$$\frac{\delta\, D(e_\otimes^F)}{\delta\varphi} = D\Big(\frac{\delta F}{\delta\varphi} \otimes e_\otimes^F\Big)\ , \tag{4.13}$$

and (under the condition Field Independence) fulfills $D(e_\otimes^{\lambda S} \otimes \varphi(h)) = \varphi(h)$, respectively.

(iii) If there are two families $R^{(m,\mu)}, \hat{R}^{(m,\mu)}$ as above, which are smooth in $m \geq 0$ and satisfy the axiom Scaling, then the corresponding intertwining family $D_H^{(m)}$ is also smooth in m, and it is independent[20] of μ and invariant under scaling:

$$\sigma_\rho D_H^{(\rho^{-1}m)}(e_\otimes^{\sigma_\rho^{-1}F}) = D_H^{(m)}(e_\otimes^F)\ . \tag{4.14}$$

[19] i.e. $D(F^{\otimes n})^* = D((F^*)^{\otimes n})$,

[20] For this reason we write $D_H^{(m)}$ instead of $D^{(m,\mu)}$.

If $R^{(m)}, \hat{R}^{(m)}$ are related to $R^{(m,\mu)}, \hat{R}^{(m,\mu)}$ by (2.37), then the corresponding intertwining $D^{(m)}$ is related to $D_H^{(m)}$ also by (2.37) and fulfills

$$\sigma_\rho \circ D_n^{(\rho^{-1}m)} \circ (\sigma_\rho^{-1})^{\otimes n} = \rho^{-\Gamma^{(m)}} \circ D_n^{(m)} \circ (\rho^{\Gamma^{(m)}})^{\otimes n} \ . \tag{4.15}$$

(iv) *Conversely, given R and D as above , equation (4.11) gives a new retarded product $\hat{R}$ with the pertinent properties. If there are families $R^{(m,\mu)}$ and $D_H^{(m)}$ as above, then the corresponding family $\hat{R}^{(m,\mu)}$ is smooth in $m \geq 0$ and satisfies the Scaling axiom.*

The identity (4.11) states that the (finite) renormalization $R \to \hat{R}$ can be absorbed in the renormalizations

- $\lambda S \to D(e_\otimes^{\lambda S})$ of the interaction and
- $F \longrightarrow D(e_\otimes^{\lambda S} \otimes F)$ of the field.

It is crucial that the renormalization of the interaction is independent of the field F. Looking in (4.11) at the terms of n-th order in λ and using the polarization identity we find that (4.11) is equivalent to

$$\begin{gathered}\hat{R}(F_1 \otimes ... \otimes F_n) = \\ \sum_{n \in I \subset \{1,...,n\}} \sum_{P \in \mathrm{Part}(I^c)} R\big(\bigotimes_{T \in P} D(F_T) \otimes D(F_I)\big)\end{gathered} \tag{4.16}$$

with $F_J = \bigotimes_{j \in J} F_j$ for an ordered index set J. It is instructive to write equation (4.16) in lowest orders:

$$\begin{aligned}
(n=1) \qquad & F \equiv \hat{R}(F) = R(D(F)) \equiv D(F), \\
(n=2) \qquad & \hat{R}(F_1 \otimes F_2) = R(F_1 \otimes F_2) + D(F_1 \otimes F_2), \\
(n=3) \qquad & \hat{R}(F_1 \otimes F_2 \otimes F_3) = R(F_1 \otimes F_2 \otimes F_3) \\
& + R(D(F_1 \otimes F_2) \otimes F_3) + R(F_1 \otimes D(F_2 \otimes F_3)) + R(F_2 \otimes D(F_1 \otimes F_3)) \\
& + D(F_1 \otimes F_2 \otimes F_3).
\end{aligned} \tag{4.17}$$

We see that the difference between the retarded products in order $(1,1)$ propagates to higher orders which gives the terms in the second last line. These terms are localized on partial diagonals and express the change of normalization of sub-diagrams. The term in the last line is localized on

the total diagonal Δ_3 and originates from the freedom of normalization of retarded products in the inductive step from $(1,1)$ to $(2,1)$.

Note that the term with $I = \{1, \ldots, n\}$ in the second line of (4.16) gives, in view of $R_{0,1} = \mathrm{id}$, a definition of $D_n \stackrel{\mathrm{def}}{=} D \restriction \mathcal{F}_{\mathrm{loc}}^{\otimes n}$ in terms of $\hat{R}_{n-1,1}$, $R_{k,1}, D_k$ for $k = 1, \ldots, n-1$. It is here that our formalism seems to be superior over previous formulations. Namely, if the retarded products take their values only on shell, $R_{0,1}$ is no longer the identity but the canonical surjection π with respect to the ideal generated by the free field equation. Then the definition of D_n requires a choice of representatives. Without the Action Ward Identity, such a choice is rather artificial, and we are not aware of any place in the literature where this problem is treated in full generality.

Proof of the Theorem. Part (iv): it is straightforward to check that every D with the properties described in the Theorem defines a new retarded product $\hat{R}$ via equation (4.11) (or equivalently (4.16)).

Part (i): from equation (4.16) one immediately sees that $\hat{R}_{0,1} = \mathrm{id}$ is equivalent to $D_1 = \mathrm{id}$, and that $\hat{R}_{n-1,1}$ is determined by the D_l's of order $l \leq n$ and by R. Vice versa, D_n is uniquely given in terms of R, $\hat{R}$ and the lower order D's, and obviously it is linear. If D satisfies the properties mentioned in Part (i) (or Parts (i)-(iii) respectively), then this holds true also for its truncation $D^{(n)}$, which is defined by $D_l^{(n)} = D_l$ for $l \leq n$ and $D_l^{(n)} = 0$ for $l > n$. Following Part (iv), $D^{(n)}$ determines a retarded product $\hat{R}^{(n)}$ with the pertinent properties, which coincides with $\hat{R}$ in order $(k,1)$ for $k < n$. From (4.16) we see that

$$D_n = \hat{R}_{n-1,1} - \hat{R}_{n-1,1}^{(n-1)}, \tag{4.18}$$

i.e. D_n is the difference between two possible extensions of retarded product at order $(n-1,1)$ and is therefore symmetric and localized on the diagonal. The latter implies ran $D_n \subset \mathcal{F}_{\mathrm{loc}}$.

Parts (ii)-(iii): additional properties of the retarded products imply directly the corresponding properties of D_n. In particular,

(Field Equation) we know that the Field Equation (3.18) determines $R_{n-1,1}(\ldots, \varphi(h))$ *uniquely* in terms of $R_{n-2,1}$; with that (4.18) implies $D_n(\ldots \otimes \varphi(h)) = 0$;

(Scaling) from (4.18) and the almost homogeneous scaling of $\hat{R}_{n-1,1}$ and $\hat{R}_{n-1,1}^{(n)}$ we conclude that $\omega_0(D_{H\,n}^{(m)}(A_1(h_1) \otimes \ldots A_n(h_n))$ must be of the form

(3.16) for $A_1,\dots A_n \in \mathcal{P}_{\rm bal}\cap\mathcal{P}_{\rm hom}$ and hence it scales homogeneously with degree $\sum_i \dim(A_i)$; this yields (4.14) for $D^{(m)}_{H\,n}$. The corresponding coefficients $C_{l,a}$ in (3.16) are independent of μ because they are dimensionless; this shows explicitly that $D^{(m)}_H$ is independent of μ. Finally (4.15) is obtained from (4.14) analogously to (2.49).

□

We now want to get a more explicit expression for $D^{(m)}_H$ under the assumptions Smoothness in m and Scaling.

Proposition 9. *Let $(D^{(m)}_H)_{m\geq 0}$ be a family of symmetric, linear maps from $T\mathcal{F}_{\rm loc}\longrightarrow \mathcal{F}_{\rm loc}$ which are local (in the sense of (4.12)) and independent of φ (4.13). Assume that the family is scale invariant and smooth in m. Then it admits the expansion*

$$D^{(m)}_{H\,n}(A_1(h_1)\otimes\cdots\otimes A_n(h_n)) = \sum_{l\in\mathbb{N}_0,a\in(\mathbb{N}_0^d)^n} m^l d_{n,l,a}(A_1\otimes\cdots\otimes A_n)(\prod_{i=1}^n \partial^{a_i}h_i) \tag{4.19}$$

with $A_i\in\mathcal{P}_{bal}$ and $h_i\in\mathcal{D}(\mathbb{M})$, where $d_{n,l,a}$ are linear symmetric maps $\mathcal{P}_{bal}^{\otimes n}\to\mathcal{P}_{bal}$ which are homogeneous in the sense that tensor products of homogeneous fields are mapped onto homogeneous fields such that the mass dimensions satisfy the relation

$$\dim(d_{n,l,a}(A_1\otimes\cdots\otimes A_n)) = \sum_{i=1}^n \dim(A_i) - l - |a| - d(n-1)\ . \tag{4.20}$$

In particular, $d_{n,l,a}$ vanishes on tensor products of fields A_i, if the right hand side is negative. Hence the sum in (4.19) is finite.

Note that we have on the right side of (4.19) the pointwise product of the test functions: $\prod_i(\partial^{a_i}h_i(x))$.

Remarks: (1) In massless models solely the terms with $l=0$ contribute in (4.19).
(2) If our requirements Smoothness in $m\geq 0$ and Scaling are replaced by the upper bound (3.7) on the scaling degree for a fixed mass m, then there is an analogous expansion, $D_n(A_1(h_1)\otimes\cdots)=\sum_a d_{n,a}(A_1\otimes\cdots)(\prod_i\partial^{a_i}h_i)$, in terms of linear symmetric maps $d_{n,a}$ which are no longer homogeneous, but

still satisfy the bound $\dim(d_{n,a}(A_1\otimes\cdots))\leq\sum_i\dim(A_i)-|a|-d(n-1)$. This change of the axioms causes only little and obvious modifications of the applications given in Sect. 5.

Proof. By the field independence $D^{(m)}_{H\,n}$ admits a Taylor expansion in φ where the coefficients are vacuum expectation values of functional derivatives of its entries (analogously to (2.35)). Because of locality (4.12) the coefficients are supported on the total diagonal and are thus derivatives of the δ-distribution in the relative coordinates. Smoothness in m implies the existence of a Taylor expansion in m around $m=0$. By integrating out the δ-distribution, reordering the sums and partial integration we can write $D^{(m)}_{H\,n}$ in the form (4.19) with $d_{n,l,a}(A_1\ldots)\in\mathcal{P}_{\rm bal}$. Since $D^{(m)}_{H\,n}$ is symmetric and multi-linear in the fields $A_1,\ldots,A_n$, the $d_{n,l,a}$'s must satisfy the corresponding properties.

The homogeneous scaling (4.14) implies $d_{n,l,a}(A_1\ldots)\in\mathcal{P}_{\rm hom}$ for $A_1,\ldots\in\mathcal{P}_{\rm hom}$; and by using (2.44), which can equivalently be written as

$$\sigma_\rho^{-1}\,A(h)=\rho^{\dim(A)}\,A(h^{(\rho)})\quad\text{with}\quad h^{(\rho)}(x)\stackrel{\rm def}{=}\rho^{-d}\,h(\rho^{-1}x)\ ,\tag{4.21}$$

we obtain

$$\begin{aligned}&\sigma_\rho\,D^{(\rho^{-1}m)}_{H\,n}(\sigma_\rho^{-1}\,A_1(h_1)\otimes\cdots)=\rho^{\sum_i\dim(A_i)}\sigma_\rho\,D^{(\rho^{-1}m)}_{H\,n}(A_1(h_1^{(\rho)})\otimes\cdots)\\&\quad=\rho^{\sum_i\dim(A_i)}\sum_{l,a}\Big(\frac{m}{\rho}\Big)^l\int dx\,(\sigma_\rho\,d_{n,l,a}(A_1\otimes\cdots)(x))(\prod_i\partial^{a_i}h_i^{(\rho)})(x)\\&=\sum_{l,a}\rho^{\sum_i\dim(A_i)-l+d-\dim(d_{n,l,a}(\ldots))-dn-|a|}m^l\int dy\,d_{n,l,a}(\ldots)(y)(\prod_i\partial^{a_i}h_i)(y)\ ,\end{aligned}\tag{4.22}$$

where we have set $y\equiv\rho^{-1}x$. Since this expression agrees with $D^{(m)}_{H\,n}(A_1(h_1)\otimes\cdots)$ the exponent of ρ must vanish; this yields (4.20). □

Remark: It is instructive to formulate the Theorem for the S-matrix as the generating functional of the time ordered products: $\mathbf{S}(\lambda S)=T(e_\otimes^{i\lambda S})$. Let R, $\hat{R}$ and D as given in the Theorem. Then the corresponding time ordered products T and $\hat{T}$ according to (E.3) and the associated S-matrices are related by

$$\hat{\mathbf{S}}(\lambda S)\equiv\hat{T}(e_\otimes^{i\lambda S})=T\Big(e_\otimes^{iD(e_\otimes^{\lambda S})}\Big)\equiv\mathbf{S}(D(e_\otimes^{\lambda S}))\ .\tag{4.23}$$

By linearity the map D of Theorem 8 can be extended to formal power series, i.e. to a map $T(\mathcal{F}_{\rm loc})[[\lambda]] \rightarrow \mathcal{F}_{\rm loc}[[\lambda]]$. Let $S(\lambda) \in \mathcal{F}_{\rm loc}[[\lambda]]$ with $S(0) = 0$. The renormalization of the interaction can be considered as a bijective analytic map

$$S(\lambda) \longrightarrow Z(S(\lambda)) \stackrel{\rm def}{=} D(e_\otimes^{S(\lambda)}) \ . \tag{4.24}$$

(When the $R^{(m,\mu)}$-products are meant we write $Z_H^{(m)}(\cdot) := D_H^{(m)}(e_\otimes^{\cdot})$.) With that admissible renormalizations of the interaction can be composed and give again an admissible renormalization. In detail, let $Z_{l,j}(\cdot) = D_{l,j}(e_\otimes^{\cdot})$, $(l, j) = (1, 2), (2, 3)$ be given, with $D_{l,j}$ satisfying the properties of the Theorem. Moreover let $\mathbf{S}_1$ be an S-matrix fulfilling the axioms. Then, $\mathbf{S}_2(S) \stackrel{\rm def}{=} \mathbf{S}_1(Z_{1,2}(S))$ and $\mathbf{S}_3(S) \stackrel{\rm def}{=} \mathbf{S}_2(Z_{2,3}(S)) = \mathbf{S}_1(Z_{1,2}(Z_{2,3}(S)))$ satisfy also the axioms (due to part (iv) of the Theorem) and, hence, $Z_{1,2} \circ Z_{2,3}$ is a renormalization of the interaction with the properties given in parts (i)-(iii) of the Theorem. So we infer that the non-uniqueness in the construction of retarded products is governed by a group, which we may call the *'Stueckelberg-Petermann renormalization group'* $\mathcal{R}$. It is the group of all analytic bijections Z of the space of formal power series $S(\lambda)$ with values in $\mathcal{F}_{\rm loc}$ and with $S(0) = 0$ (thus $Z(0) = 0$), which satisfy the conditions

(i) Z preserves the first order term, i.e.

$$\frac{d}{d\lambda} Z(S(\lambda))|_{\lambda=0} = \frac{d}{d\lambda} S(\lambda)|_{\lambda=0} \ .$$

(ii) Z is real, i.e. $Z(S(\lambda)^*) = Z(S(\lambda))^*$.

(iii) Z is local: it preserves the localization region,

$$\text{supp}\, \frac{\delta\, Z(S(\lambda))}{\delta\varphi} = \text{supp}\, \frac{\delta\, S(\lambda)}{\delta\varphi} \ ,$$

and it is additive on sums of terms with disjoint localizations,

$$Z(S_1(\lambda) + S_2(\lambda)) = Z(S_1(\lambda)) + Z(S_2(\lambda))$$

if $\text{supp}\, \frac{\delta\, S_1(\lambda)}{\delta\varphi} \cap \text{supp}\, \frac{\delta\, S_2(\lambda)}{\delta\varphi} = \emptyset$.

(iv) Z is Poincaré invariant.

(v) Z does not explicitely depend on φ.

(vi) Z acts trivially on φ in the sense that

$$Z(S(\lambda) + \lambda\varphi(h)) = Z(S(\lambda)) + \lambda\varphi(h) \ .$$

(vii) In the preceding conditions it is not assumed that the retarded products satisfy the axioms Smoothness and Scaling. If the latter are included, Z is replaced by a family $Z_H = (Z_H^{(m)})_{m\geq 0}$, where each component $Z_H^{(m)}$ must fulfill (i)-(vi) and additionally it must hold

$$Z_H^{(\rho m)} = \sigma_\rho \circ Z_H^{(m)} \circ \sigma_{\rho^{-1}}$$

and that $Z_H^{(m)}$ is smooth in $m \geq 0$.

In terms of Z and its first derivative Z' the transformation formula (4.11) of the retarded products reads

$$\hat{F}_S = (Z'(S)F)_{Z(S)} \tag{4.25}$$

where

$$Z'(S)F := \frac{d}{d\tau} Z(S + \tau F)|_{\tau=0} \ . \tag{4.26}$$

5 The algebraic adiabatic limit and the renormalization group

Of particular interest are certain factor groups of the Stueckelberg-Petermann renormalization group $\mathcal{R}$ introduced in the preceding section. Up to now our renormalization group transformations act on explicitely spacetime dependent interaction Lagrangians $g = \sum g_i \mathcal{L}_i \in \mathcal{D}(\mathbb{M}, \mathcal{P}_{\mathrm{bal}})$. We want to extract from this information the action of the renormalization group on constant Lagrangians $\mathcal{L} \in \mathcal{P}_{\mathrm{bal}}$. This requires a kind of adiabatic limit. The usual adiabatic limit $g \to k$ (where k is constant) needs a good infrared behavior (see e.g. [18, 19, 2, 40]). We therefore work in the algebraic adiabatic limit.

To explain the **algebraic adiabatic limit** let $\mathcal{O} \subset \mathbb{M}$ be a causally closed, open region. Let $S \in \mathcal{F}_{\mathrm{loc}}$. Due to Proposition 2 there is a unique function $g \in \mathcal{D}(\mathbb{M}, \mathcal{P}_{\mathrm{bal}})$ with $\int g = S$; we may therefore write $A(h)_g$ instead of $A(h)_S$ for the interacting fields and $Z(g)$, $Z'(g)$ for $Z(\int g)$ (4.24) and $Z'(\int g)$ (4.26). We introduce the algebra $\mathcal{A}_g(\mathcal{O})$ of interacting fields belonging to $\mathcal{O}$ as the sub-algebra of $\mathcal{A}^{(m)}$ which is generated by the interacting

fields $A(h)_g$ with $A \in \mathcal{P}$ and $h \in \mathcal{D}(\mathcal{O})$. Note that $\mathcal{A}_g(\mathcal{O})$ depends on the chosen retarded products. For $\mathcal{L} \in \mathcal{P}_{\rm bal}$ we define

$$\begin{aligned}\mathcal{G}_{\mathcal{L}}(\mathcal{O}) \stackrel{\rm def}{=} & \{g \in \mathcal{D}(\mathbb{M}, \mathcal{P}_{\rm bal}) \mid \\ & g(x) = \mathcal{L} \text{ for all } x \text{ in a neighborhood of the closure of } \mathcal{O}\} \ .\end{aligned}$$

The algebraic adiabatic limit relies on the following observation [7][21]: for any $g_1, g_2 \in \mathcal{G}_{\mathcal{L}}(\mathcal{O})$ there exists a set Aut_{g_1,g_2} of automorphisms α of $\mathcal{A}^{(m)}[[\lambda]]$ with

$$\alpha\big(A(h)_{g_1}\big) = A(h)_{g_2} \quad \forall \alpha \in \mathrm{Aut}_{g_1,g_2},\ \forall h \in \mathcal{D}(\mathcal{O}),\ \forall A \in \mathcal{P}\ . \tag{5.1}$$

This has the important consequence that the *algebraic structure of* $\mathcal{A}_g(\mathcal{O})$ *is independent of the choice of* $g \in \mathcal{G}_{\mathcal{L}}(\mathcal{O})$.

Following [7, 13] we may formalize the algebraic adiabatic limit in the following way: Consider the bundle of algebras $\mathcal{A}_g(\mathcal{O})$ over the space of compactly supported Lagrangians $g \in \mathcal{G}_{\mathcal{L}}(\mathcal{O})$ where $\mathcal{L} \in \mathcal{P}_{\rm bal}$ is a constant Lagrangian.

A section $B = (B_g)$ is called covariantly constant if it holds

$$\alpha(B_{g_1}) = B_{g_2} \quad \forall \alpha \in \mathrm{Aut}_{g_1,g_2}\ .$$

In particular the interacting fields are covariantly constant sections. The local algebra $\mathcal{A}_{\mathcal{L}}(\mathcal{O})$ is now defined as the algebra of covariantly constant sections of the bundle introduced above.

To get a net of local algebras in the sense of the Haag-Kastler axioms [27] one has in addition to fix the embeddings into algebras of larger regions. Let $\mathcal{O}_1 \subset \mathcal{O}_2$. Then we define the embeddings of algebras

$$i_{\mathcal{O}_2\mathcal{O}_1} : \mathcal{A}_{\mathcal{L}}(\mathcal{O}_1) \to \mathcal{A}_{\mathcal{L}}(\mathcal{O}_2)$$

by restricting a section B to Lagrangians $g \in \mathcal{G}_{\mathcal{L}}(\mathcal{O}_2)$.

It is easy to see that these embeddings satisfy the consistency condition

$$i_{\mathcal{O}_3\mathcal{O}_2} \circ i_{\mathcal{O}_2\mathcal{O}_1} = i_{\mathcal{O}_3\mathcal{O}_1}$$

for $\mathcal{O}_1 \subset \mathcal{O}_2 \subset \mathcal{O}_3$. Moreover, the net is covariant under Poincaré transformations provided the Lagrangian $\mathcal{L}$ is Lorentz invariant [13]. It also satisfies

[21] An alternative proof of (5.1) which is based on our axioms for retarded products (Sect. 2) is given in [14]. It deals with on-shell valued retarded products; however it applies also to our $\mathcal{F}$-valued retarded products because it uses the Symmetry, Causality and GLZ relation only.

the condition of local commutativity as a consequence of the conditions GLZ relation and Causality.

We may also look for local fields associated to the net. By definition, a local field associated to the net is a family of distributions $(A_{\mathcal{O}})$ with values in $\mathcal{A}_{\mathcal{L}}(\mathcal{O})$ such that

$$i_{\mathcal{O}_2\mathcal{O}_1}(A_{\mathcal{O}_1}(h)) = A_{\mathcal{O}_2}(h)$$

if supp $h \subset \mathcal{O}_1$ and which transform covariantly under Poincaré transformations [8]. Examples for local fields are given in terms of the classical fields $A \in \mathcal{P}$ by the sections

$$(A_{\mathcal{O}}(h))_g := (A(h))_g \ , \ g \in \mathcal{G}_{\mathcal{L}}(\mathcal{O}) \ .$$

It is an open question whether there are other local fields. This amounts to a determination of the Borchers class for perturbatively defined interacting field theories.

We are now going to investigate what happens with the Stueckelberg-Petermann renormalization group $\mathcal{R}$ in the algebraic adiabatic limit. For this purpose we insert a $g \in \mathcal{G}_{\mathcal{L}}(\mathcal{O})$ into (4.19) and find

$$Z_H(\lambda g) = D_H(e_\otimes^{\lambda \int g}) \in \mathcal{G}_{z(\lambda\mathcal{L})}(\mathcal{O}) \tag{5.2}$$

with

$$z(A) = \sum_{n,l} \frac{1}{n!} m^l d_{n,l,0}(A^{\otimes n}) \ , \quad A \in \lambda\mathcal{P}_{\mathrm{bal}}[[\lambda]] \ . \tag{5.3}$$

(The terms $a \neq 0$ with derivatives of the test functions in (4.19) do not contribute to z because $g|_{\mathcal{O}} =$ constant.) Hence, the renormalization group transformations $Z_H \in \mathcal{R}$ induce transformations

$$z : \lambda\mathcal{P}_{\mathrm{bal}}[[\lambda]] \longrightarrow \lambda\mathcal{P}_{\mathrm{bal}}[[\lambda]] \ : \ z(\lambda\mathcal{L}) = \lambda\mathcal{L} + \mathcal{O}(\lambda^2). \tag{5.4}$$

Since Z_H is invertible this holds also for z. The map $\gamma : Z_H \mapsto z$ is a homomorphism of $\mathcal{R}$ to the *renormalization group in the adiabatic limit* $\mathcal{R}_{\mathrm{adlim}} := \gamma(\mathcal{R})$. As one might expect, the kernel of this homomorphism acts trivially on the local nets. Actually, this holds already in case the given Lagrangian is left invariant, in detail:

Theorem 10. *Let $\mathcal{A}_{\mathcal{L}}$ and $\hat{\mathcal{A}}_{\mathcal{L}}$ be two local nets which are defined by renormalization prescriptions $R^{(m,\mu)}$ and $\hat{R}^{(m,\mu)}$ which are related by a renormalization group transformation Z_H (4.24) such that $\gamma(Z_H)(\lambda\mathcal{L}) = \lambda\mathcal{L}$. Then the nets are equivalent, i.e. there exist isomorphisms $\beta_{\mathcal{O}} : \mathcal{A}_{\mathcal{L}}(\mathcal{O}) \to \hat{\mathcal{A}}_{\mathcal{L}}(\mathcal{O})$ with $\beta_{\mathcal{O}_2} \circ i_{\mathcal{O}_2\mathcal{O}_1} = \hat{i}_{\mathcal{O}_2\mathcal{O}_1} \circ \beta_{\mathcal{O}_1}$.*

Proof. Since $\hat{F}_g = (Z'_H(g)F)_{Z_H(g)}$ (4.25) where $Z'_H(g)$ is an invertible linear transformation on the space of local functionals, the algebras $\hat{\mathcal{A}}_g(\mathcal{O})$ and $\mathcal{A}_{Z_H(g)}(\mathcal{O})$ coincide. In addition note that $g \in \mathcal{G}_{\mathcal{L}}(\mathcal{O})$ is equivalent to $Z_H(g) \in \mathcal{G}_{\mathcal{L}}(\mathcal{O})$. We are now going to show that a section B in $\mathcal{A}_{\mathcal{L}}(\mathcal{O})$ can be mapped to a section $\beta_{\mathcal{O}}(B)$ in $\hat{\mathcal{A}}_{\mathcal{L}}(\mathcal{O})$ by

$$\beta_{\mathcal{O}}(B)_g = B_{Z_H(g)} \ .$$

Obviously, this map is an isomorphism of the algebras of sections. It remains to prove, that B is covariantly constant if and only if $\beta_{\mathcal{O}}(B)$ is. But this follows from the fact that $Z'_H(g)F = D_H(e_{\otimes}^{\int g} \otimes F)$ is independent of the choice of $g \in \mathcal{G}_{\mathcal{L}}(\mathcal{O})$ if F is localized within $\mathcal{O}$. Therefore the conditions on the intertwining isomorphisms α are identical: $\widehat{\mathrm{Aut}}_{g_1,g_2}(\mathcal{O}) = \mathrm{Aut}_{Z_H(g_1),Z_H(g_2)}(\mathcal{O})$. $\square$

Due to this Theorem, $\gamma(Z_H) = z$ is the **renormalization of the interaction in the algebraic adiabatic limit**. From (4.20) and (5.3) we immediately find

$$\dim(\mathcal{L}) \leq d \quad \Rightarrow \quad \dim(z(\lambda\mathcal{L})) \leq d \ . \tag{5.5}$$

Example: **renormalization of $\mathcal{L} = \lambda\varphi^4$ in $d = 4$ dimensions**. In case $R^{(m,\mu)}$ and $\hat{R}^{(m,\mu)}$ respect the field parity (2.10), this holds also for the corresponding map D_H of Theorem 8 and hence for z (5.3). Taking this, Lorentz covariance, Unitarity, (5.4) and (5.5) into account we obtain[22]

$$z(\lambda\varphi^4) = \lambda\big[(1+a)\varphi^4 + b((\partial\varphi)^2 - \varphi\Box\varphi) + m^2 c\varphi^2 + m^4 e\mathbf{1}\big] \ , \tag{5.6}$$

where $a, b, c, e \in \lambda\mathbb{R}[[\lambda]]$. The term $m^4 e\mathbf{1}$ is irrelevant because $R^{(m,\mu)}_{n,1}(\ldots\mathbf{1}\ldots) = 0$ for $n \geq 1$ (see e.g. [12], Lemma 1(C)). a describes coupling constant renormalization, b and c wave function and mass renormalization. As usual, the latter renormalizations can be absorbed in a redefinition of the free theory, so that there is only one free parameter left.

Remark: The invariance of the Lagrangian is sufficient but not necessary for the invariance of the net (in the example (5.6) the parameter e is irrelevant for

[22] According to (3.4) there is (up to multiplication with a constant) precisely one Lorentz invariant balanced field with two factors φ and two derivatives, namely $\partial^{12\mu}\partial^{12}_{\mu}\varphi(x_1)\varphi(x_2)|_{x_1=x_2=x} = 2(\varphi\Box\varphi - (\partial^{\mu}\varphi)\partial_{\mu}\varphi)(x)$.

the structure of the net). We plan to determine the corresponding subgroup of $\mathcal{R}_{\rm adlim}$ [6].

The **field renormalization in the algebraic adiabatic limit** is a map

$$z^{(1)}:\lambda\mathcal{P}_{\rm bal}[[\lambda]]\times\mathcal{P}_{\rm bal}\longrightarrow\mathcal{P}[[\lambda]]\;:\;(\lambda\mathcal{L},A)\mapsto z^{(1)}(\lambda\mathcal{L})A=A+\mathcal{O}(\lambda)\quad(5.7)$$

which is determined by the requirement that the transformation formula (4.11) or (4.25) takes the simpler form

$$(z^{(1)}(\lambda\mathcal{L})A)(h)_{z(\lambda\mathcal{L})}=\hat{A}(h)_{\lambda\mathcal{L}}\;,\quad\forall h\in\mathcal{D},\,\forall A,\mathcal{L}\in\mathcal{P}_{\rm bal}\;,\qquad(5.8)$$

in the algebraic adiabatic limit. This condition has a unique solution: in the formula (4.19) for $D_{H\,n}((\lambda\int g)^{\otimes n}\otimes A(h))$ we specialize to $g\in\mathcal{G}_{\mathcal{L}}(\mathcal{O})$ and $h\in\mathcal{D}(\mathcal{O})$ and perform partial integrations. This yields

$$z^{(1)}(\lambda\mathcal{L})A=\sum_{n,l,a\in\mathbb{N}_0^d}\frac{1}{n!}m^l(-1)^{|a|}\partial^a d_{n+1,l,(0,\dots,0,a)}((\lambda\mathcal{L})^{\otimes n}\otimes A)\;.\qquad(5.9)$$

Hence, the algebraic adiabatic limit simplifies the field renormalization $A(h)\mapsto D_H(e_\otimes^{\lambda\int g}\otimes A(h))$ to the **linear** map $\mathcal{P}_{\rm bal}\ni A\mapsto z_1(\lambda\mathcal{L})A\in\mathcal{P}[[\lambda]]$, i.e. the test function remains unchanged. By using the definition

$$z^{(1)}(\lambda\mathcal{L})\partial^a A:=\partial^a z^{(1)}(\lambda\mathcal{L})A\;,\quad A\in\mathcal{P}_{\rm bal}\;,\qquad(5.10)$$

and linearity we extend $z^{(1)}$ to a map $\mathcal{P}_{\rm bal}[[\lambda]]\times\mathcal{P}\to\mathcal{P}[[\lambda]]$ and with that the relation (5.8) holds even for all $A\in\mathcal{P}$. From (4.20) and (5.9) we conclude

$$\dim(\mathcal{L})\leq d\quad\Rightarrow\quad\dim(z^{(1)}(\lambda\mathcal{L})A)=\dim(A)\qquad(5.11)$$

(where d is the number of space time dimensions). In a *massless* model with $\mathcal{L}\in\mathcal{P}_d$ (2.41) we find

$$z(\lambda\mathcal{L})\in\mathcal{P}_d\quad\text{and}\quad A\in\mathcal{P}_j\Rightarrow z^{(1)}(\lambda\mathcal{L})A\in\mathcal{P}_j\;,\qquad(5.12)$$

due to Remark (1) after Proposition 9. We point out that z and $z^{(1)}$ are uniquely determined by $R^{(m,\mu)}$ and $\hat{R}^{(m,\mu)}$ and that they are universal, i.e. independent of $\mathcal{O}$.

Example: **renormalization of φ and φ^2 in a massless model with** $\mathcal{L}\in\mathcal{P}_d$. In case the retarded products satisfy the Field equation, the renormalization of φ is the identity, due to part (ii) of Theorem 8. Here we do

not use this assumption, instead we work out some consequences of (5.12). Since $\mathcal{P}_1 = \{a\varphi | a \in \mathbb{R}\}$ the renormalization of φ has the simple form

$$\varphi \longrightarrow z_\varphi^{(1)} \varphi \quad , \quad z_\varphi^{(1)} \in \mathbb{R}[[\lambda]] \ . \tag{5.13}$$

For $A = \varphi^2$ the renormalized field is of the form

$$z^{(1)}(\lambda\mathcal{L})\varphi^2 = z_0^{(1)}\varphi^2 + z_{1\mu}^{(1)}\partial^\mu\varphi \ , \qquad z_0^{(1)},\ z_{1\mu}^{(1)} \in \mathbb{R}[[\lambda]] \ . \tag{5.14}$$

In case $\mathcal{L}$, $R^{(m,\mu)}$ and $\hat{R}^{(m,\mu)}$ are Lorentz covariant, the right side of (5.14) must also have this symmetry, i.e. $z_{1\mu}^{(1)} = 0$. (Alternatively, the latter follows also if $\mathcal{L}$ is even in φ and $R^{(m,\mu)}$ and $\hat{R}^{(m,\mu)}$ preserve the field parity (2.10).) With that the renormalization of φ^2 is 'diagonal', too. Usually $z_0^{(1)}$ (5.14) is non-trivial and it cannot be related to $(z_\varphi^{(1)})^2$ (5.13).

Scaling transformations: in Sect. 3 we have shown that one can fulfill almost homogeneous scaling of $R_{n,1}^{(m,\mu)}$ for all $m \geq 0$. The results of this Section yield far reaching additional information about the connection of $R_\rho^{(m,\mu)}$ (2.49) and $R^{(m,\mu)}$ (cf. [32] and [25]). The basic observation is the following: if $R^{(m,\mu)}$ fulfills the axioms given in Sect. 2 (Lorentz covariance, global inner symmetries and the Field equation may be excluded or included), then the same axioms hold true for $R_\rho^{(m,\mu)}$, too, as can be verified straightforwardly. Therefore, there exists a sequence $D_{H\,\rho}^{(m)}$ (4.10) with the properties mentioned in the Main Theorem. So, the *scaling transformations on a given renormalization prescription induce a one parameter subgroup of the renormalization group* $\mathcal{R}_{\mathrm{adlim}}$, which may be called *'Gell-Mann-Low Renormalization Group'*.

With the scaling as renormalization transformation we are now going to compute to lowest non-trivial order the renormalization of the fields $\varphi^2, \varphi^3, \varphi^4$ and of the interaction $\lambda\mathcal{L}$ for $\mathcal{L} = \varphi^4$ (in $d = 4$ dimensions) and $m = 0$. We assume that $R \equiv R^{(0)} \equiv R^{(0,\mu)}$ fulfills all axioms of Sect. 2.

Renormalization of φ^2: by (4.16) and the expansion (2.35)-(2.36) we obtain

$$\begin{aligned} D_\rho(\varphi^4(x_1) \otimes \varphi^2(x)) &= R_\rho(\varphi^4(x_1), \varphi^2(x)) - R(\varphi^4(x_1), \varphi^2(x)) \\ &= 6\big[\rho^4 r(\varphi^2, \varphi^2)(\rho(x_1 - x)) - r(\varphi^2, \varphi^2)(x_1 - x)\big]\varphi^2(x_1) \\ &= \frac{6}{(2\pi)^2} \log\rho \ \delta(x_1 - x)\, \varphi^2(x_1) \ , \end{aligned} \tag{5.15}$$

where we use a symbolic notation for D_ρ and the result (C.7) of Appendix C. The tree diagram of $R(\varphi^4,\varphi^2)$ does not contribute, because it scales homogeneously. Going over to the algebraic adiabatic limit the formula (5.9) yields

$$z_\rho^{(1)}(\lambda\varphi^4)\varphi^2 = \Big(1+\lambda\frac{6}{(2\pi)^2}\log\rho \;+\; \mathcal{O}(\lambda^2)\Big)\varphi^2 \;, \tag{5.16}$$

independently of the normalization of the fish diagram $r(\varphi^2,\varphi^2)$.

Renormalization of φ^3: analogously to (5.15) we get

$$\begin{gathered} D_\rho(\varphi^4(x_1)\otimes\varphi^3(x)) = \\ 18\big[\rho^4 r(\varphi^2,\varphi^2)(\rho(x_1-x)) - r(\varphi^2,\varphi^2)(x_1-x)\big]\varphi(x)\varphi^2(x_1) \\ +4[\rho^6 r(\varphi^3,\varphi^3)(\rho(x_1-x)) - r(\varphi^3,\varphi^3)(x_1-x)]\varphi(x_1) = \\ +\frac{18}{(2\pi)^2}\log\rho\,\delta(x_1-x)\varphi^3(x_1) \\ +\frac{3}{2(2\pi)^4}\,(\log\rho)\,\Box\delta(x_1-x)\varphi(x_1)\;, \end{gathered} \tag{5.17}$$

where we have inserted the results (C.7) and (C.13) of Appendix C. In the algebraic adiabatic limit this gives

$$\begin{aligned} z_\rho^{(1)}(\lambda\varphi^4)\varphi^3 \;&=\; \Big(1+\lambda\frac{18}{(2\pi)^2}\log\rho+\mathcal{O}(\lambda^2)\Big)\,\varphi^3 \\ &+\; \Big(\lambda\frac{3}{2(2\pi)^4}\log\rho+\mathcal{O}(\lambda^2)\Big)\,\Box\varphi \end{aligned} \tag{5.18}$$

by means of (5.9). Other terms than φ^3 and $\Box\varphi$ do not appear to higher orders either, due to (5.12), the maintenance of the field parity and Lorentz invariance. So, the field renormalization of φ^3 is non-diagonal. However, since $z_\rho^{(1)}(\lambda\varphi^4)\Box\varphi=\Box\varphi$ (due to the validity of the Field equation and (5.10)), the field $(\varphi^3+\frac{1}{12\,(2\pi)^2}\Box\varphi)$ is an eigenvector of $z_\rho^{(1)}(\lambda\varphi^4)$ with eigenvalue $\big(1+\lambda\frac{18}{(2\pi)^2}\log\rho+\mathcal{O}(\lambda^2)\big)$.

Renormalization of $\mathcal{L}=\varphi^4$: we continue the Example (5.6) for the particular case of the scaling transformations and $m=0$. Since the corre-

sponding tree diagrams scales homogeneously we obtain

$$
\begin{aligned}
&D_\rho(\varphi^4(x_1)\otimes\varphi^4(x)) = \\
&36\Big\{\rho^4 r(\varphi^2,\varphi^2)(\rho(x_1-x)) - r(\varphi^2,\varphi^2)(x_1-x)\Big\}\varphi^2(x_1)\varphi^2(x)\\
&+16\Big\{\rho^6 r(\varphi^3,\varphi^3)(\rho(x_1-x)) - r(\varphi^3,\varphi^3)(x_1-x)\Big\}\varphi(x_1)\varphi(x)\\
&+\Big\{\rho^8 r(\varphi^4,\varphi^4)(\rho(x_1-x)) - r(\varphi^4,\varphi^4)(x_1-x)\Big\}\mathbf{1}\\
&= \frac{36}{(2\pi)^2}\,(\log\rho)\,\delta(x_1-x)\varphi^2(x_1)\varphi^2(x)\\
&+\frac{6}{(2\pi)^4}\,(\log\rho)\,(\Box\delta)(x_1-x)\,\varphi(x_1)\varphi(x)+\ldots\Box\Box\delta(x_1-x)\mathbf{1}
\end{aligned}
\tag{5.19}
$$

by using (C.7) and (C.13); the form of the last term follows from Lorentz invariance and that it must scale homogeneously with degree 8. In the algebraic adiabatic limit we get the following renormalization of the interaction:

$$
\begin{aligned}
z_\rho(\lambda\varphi^4) &= \Big(\lambda+\lambda^2\frac{18}{(2\pi)^2}\log\rho+\mathcal{O}(\lambda^3)\Big)\varphi^4\\
&+ \Big(-\lambda^2\frac{3}{2(2\pi)^4}\log\rho+\mathcal{O}(\lambda^3)\Big)\big((\partial^\mu\varphi)\partial_\mu\varphi\ -\varphi\Box\varphi\big),
\end{aligned}
\tag{5.20}
$$

where we apply (5.3) and take into account that z takes values in $\mathcal{P}_{\mathrm{bal}}$. Due to (5.12) there is no mass renormalization, i.e. the constant c in (5.6) vanishes to all orders.

Field Renormalization of φ^4: from $D_\rho(\varphi^4\otimes\varphi^4)$ (5.19) we can also read off the field renormalization of φ^4 to first order in λ:

$$
z_\rho^{(1)}(\lambda\varphi^4)\varphi^4=\varphi^4+\lambda\Big(\frac{36}{(2\pi)^2}\log\rho\,\varphi^4+\frac{6}{(2\pi)^4}\log\rho\,\varphi\Box\varphi\Big)+\mathcal{O}(\lambda^2)\ . \tag{5.21}
$$

6 Outlook

The construction of a renormalized perturbative quantum field theory, in the sense of algebraic quantum field theory [27], was carried through without ever meeting infrared problems. In particular, the renormalization group (in the sense of Stueckelberg and Petermann) could be constructed in purely local terms. This in variance with standard techniques of perturbation theory which typically rely on global properties.

Given the algebra of interacting fields, one may then, in a second step, look for states of interest, for instance vacuum or particle states. This amounts to perform the adiabatic limit in the conventional sense and was done for massive theories by Epstein and Glaser [18, 19] and, on the basis of retarded products, by Steinmann [42]. (For QED see [2] for the construction of the vacuum state and [43] for the analysis of scattering.) One then may relate the global renormalization parameters, as masses and coupling constants at e.g. zero momentum, to the local parameters involved in our construction. One may also look for other situations, for instance at finite temperature or with non-trivial boundary conditions. Then, other global parameters are of interest, but the local parameters remain the same.

On a generic curved space-time it seems that a *completely local procedure* is by far the best way to construct perturbative quantum fields; in particular the large ambiguity of renormalization in theories without translation invariance has recently been removed (up to few parameters) by requiring the generally covariant locality principle [31, 32, 8].

In connection with the renormalization group the following topics will be studied in a subsequent paper [6].

- The map D of the Main Theorem (Theorem 8) scales homogeneously for the *modified* interacting fields only. For this reason most applications of this Theorem given in Sects. 4 and 5 are restricted to the modified interacting fields. These results can be translated into statements about the original interacting fields by the transformation formula (2.37) (which holds also for $(D^{(m)}, D^{(m,\mu)})$).

- The generator of the Gell-Mann-Low Renormalization Group (i.e. the subgroup of $\mathcal{R}_{\mathrm{adlim}}$ induced by the scaling transformations on a given renormalization prescription R) is related to the β function. The Gell-Mann-Low subgroups belonging to different renormalization prescriptions R and $\hat{R}$ are conjugate to each other. The generator starts with a term of second order which is universal.

- The absorption of the b- and c- term of (5.6) in a redefinition of the free theory (wave function and mass renormalization) requires that the *physical predictions are independent of the splitting of the action in a free and an interacting part* (where the free part is always quadratic in $\partial^a\varphi$ $(a \in \mathbb{N}_0^d)$). It turns out that the latter is an additional (re)normalization

condition, which is part of the 'Principle of Perturbative Agreement' required by Hollands and Wald [33].

- The scaling transformations are the bridge to Wilson's renormalization group. This has to be investigated as well as the connection to the Buchholz-Verch scaling limit.

- Hollands and Wald made a corresponding analysis for curved space-times [32]. But the present formalism is not yet fully adapted to general Lorentzian space-times.

Appendices

A Mass dependence of the two-point function

To investigate the mass dependence of the two-point function $\Delta_m^{+(d)}(x_1 - x_2) \equiv \omega_0(\varphi(x_1) \star_m \varphi(x_2))$ in d-dimensions we compute the Fourier transformation[23]

$$\Delta_m^{+(d)}(y) = \frac{1}{(2\pi)^{d-1}} \int d^dp\, \Theta(p^0)\delta(p^2 - m^2)e^{-ipy} \tag{A.1}$$

in the sense of distributions. We perform the p^0-integration and use

$$\int d^{d-1}\vec{p} \ldots = |S_{d-3}| \int_0^\infty dp\, p^{d-2} \int_0^\pi d\theta\, (\sin\theta)^{d-3} \ldots\ , \tag{A.2}$$

where

$$|S_k| = \frac{2\pi^{\frac{k+1}{2}}}{\Gamma(\frac{k+1}{2})} \tag{A.3}$$

is the surface of the unit ball in $\mathbb{R}^{k+1}$. With $y \equiv (t, \vec{y})$, $r \equiv |\vec{y}|$ this gives

$$\Delta_m^{+(d)}(y) = \frac{|S_{d-3}|}{(2\pi)^{d-1}} \int_0^\infty dp\, p^{d-2} \int_0^\pi d\theta\, (\sin\theta)^{d-3}\, e^{ipr\cos\theta}\, \frac{e^{-i\omega t}}{2\omega}|_{\omega=\sqrt{p^2+m^2}}\ . \tag{A.4}$$

It is well known that $\Delta_m^{+(d)}$ is the limit of a function which is analytic in the forward tube $\mathbb{R}^d - iV_+^{(d)}$. (This is e.g. a consequence of the Wightman axioms.) Taking additionally Lorentz covariance into account we conclude that $\Delta_m^{+(d)}$ is of the form

$$\Delta_m^{+(d)}(y) = \lim_{\epsilon\to 0} f(y^2 - iy^0\epsilon)\ , \tag{A.5}$$

where $f(z)$ is analytic for $z \in \mathbb{C} \setminus U$ for some $U \subset \mathbb{R}$. With that it suffices to compute $\Delta_m^{+(d)}$ for $y = (t, \vec{0})$, $t > 0$. Namely, for $d = 3$ we obtain

$$\begin{aligned}\Delta_m^{+(3)}(t, \vec{0}) &= \frac{1}{4\pi} \int_0^\infty dp\, p\, \frac{e^{-i\omega t}}{\omega}|_{\omega=\sqrt{p^2+m^2}} = \frac{i}{4\pi}\frac{\partial}{\partial t} \int_0^\infty dp\, p\, \frac{e^{-i\omega t}}{\omega^2} \\ &= \frac{i}{4\pi}\frac{\partial}{\partial t} \int_{mt}^\infty du\, \frac{e^{-iu}}{u} = \frac{-i}{4\pi}\, \frac{e^{-imt}}{t}\ .\end{aligned} \tag{A.6}$$

[23] An analogous (unpublished) computation of the commutator function by K.-H. Rehren and M. D. was very helpful for writing essential parts of this Appendix.

Due to (A.5) $\Delta_m^{+(3)}(y)$ is obtained for arbitrary y by replacing it (in the latter formula) by $\sqrt{-(y^2 - iy^0 0)}$. This gives

$$\Delta_m^{+(3)}(y) = \frac{1}{4\pi \sqrt{-(y^2 - iy^0 0)}}\, e^{-m\sqrt{-(y^2 - iy^0 0)}} \ . \tag{A.7}$$

Analogously, for $d = 4$ one obtains

$$\Delta_m^{+(4)}(y) = \frac{-1}{4\pi^2 (y^2 - iy^0 0)} + \log(-m^2(y^2 - iy^0 0))\, m^2\, f(m^2 y^2) + m^2 F(m^2 y^2) \ , \tag{A.8}$$

where f and F are analytic functions, see e.g. Sect. 15.1 of [3]. f can be expressed in terms of the Bessel function J_1 of order 1, namely

$$f(z) \equiv \frac{1}{8\pi^2 \sqrt{z}}\, J_1(\sqrt{z}) = \sum_{k=0}^{\infty} C_k\, z^k \ , \quad C_k \in \mathbb{R} \ ; \tag{A.9}$$

and F is given by a power series

$$F(z) \equiv -\frac{1}{4\pi} \sum_{k=0}^{\infty} \{\psi(k+1) + \psi(k+2)\} \frac{(-z/4)^k}{k!(k+1)!} \ , \tag{A.10}$$

where the Psi-function is related to the Gamma-function by $\psi(x) \equiv \Gamma'(x)\,/\,\Gamma(x)$.

We see that $\Delta_m^{+(3)}$ is smooth in $m \geq 0$, but $\Delta_m^{+(4)}$ is not smooth at $m = 0$ (it is only continuously differentiable)! However,

$$H_m^{\mu\,(4)}(y) \equiv \Delta_m^{+(4)}(y) - m^2\, f(m^2 y^2)\log(m^2/\mu^2) \tag{A.11}$$

(where $\mu > 0$ is a **fixed** mass-parameter) is smooth in $m \geq 0$. In addition,

- $H_m^{\mu\,(4)}(y) - \Delta_m^{+(4)}(y)$ is a smooth function of y, i.e. the wave front sets of $H_m^{\mu\,(4)}(y)$ and $\Delta_m^{+(4)}(y)$ agree and, hence, $(H_m^{\mu\,(4)}(y))^k$, $k \in \mathbb{N}$ exists;
- the antisymmetric part of $H_m^{\mu\,(4)}$ is the same as for $\Delta_m^{+(4)}$ (namely $= i\Delta_m^{(4)}/2$, where $\Delta_m^{(d)}$ is the commutator function);
- $H_m^{\mu\,(4)}$ is Poincaré invariant;
- $H_m^{\mu\,(4)}$ satisfies the Klein-Gordon equation since $(\Box_y + m^2)\, f(m^2\, y^2) = 0$ (by using Bessel's differential equation);

- $H^{\mu\,(4)}_m$ does not scale homogeneously:

$$\rho^2\, H^{\mu(4)}_{\rho^{-1}m}(\rho y) - H^{\mu(4)}_m(y) = \log(\rho)\, 2m^2\, f(m^2y^2)\ ; \tag{A.12}$$

- and $H^{\mu\,(4)}_{m=0} = \Delta^{+(4)}_{m=0}$.

To investigate the mass dependence of the two-point function in dimensions $d \geq 5$ we derive a recursion relation which expresses $\Delta^{+(d+2)}_m$ in terms of $\Delta^{+(d)}_m$. From (A.4) we find

$$\begin{aligned}(\partial_r^2 - \partial_t^2 - m^2)\Delta^{+(d)}_m(y) &= |S_{d-3}| \int_0^\infty dp\, p^d \int_0^\pi d\theta\, (\sin\theta)^{d-1} e^{-ipr\cos\theta} \frac{e^{i\omega t}}{2\omega} \\ &= (2\pi)^2\, \frac{|S_{d-3}|}{|S_{d-1}|}\, \Delta^{+(d+2)}_m(y)\ . \end{aligned} \tag{A.13}$$

By using $\Box^{(d)}_y = (\partial_t^2 - \partial_r^2 - \frac{d-2}{r}\partial_r +$ derivatives with respect to the angles of $\vec{y}$) (the latter vanish in $\Box^{(d)}\Delta^{+(d)}_m$) and $(\Box^{(d)} + m^2)\Delta^{+(d)}_m = 0$ we obtain

$$\Delta^{+(d+2)}_m(y) = \frac{-1}{2\pi r}\, \partial_r\, \Delta^{+(d)}_m(y)\ . \tag{A.14}$$

Because of $\rho^{d-2}\Delta^{+(d)}_{\rho^{-1}m}(\rho y) = \Delta^{+(d)}_m(y)$ and Poincaré invariance, $\Delta^{+(d)}_m$ is of the form

$$\Delta^{+(d)}_m(y) = m^{d-2}\, F^{(d)}(m^2\, (y^2 - iy^0 0))\ . \tag{A.15}$$

With that we obtain

$$\Delta^{+(d+2)}_m(y) = \frac{1}{2\pi(y^2 - iy^0 0)}\, (m\partial_m + 2 - d)\, \Delta^{+(d)}_m(y)\ . \tag{A.16}$$

The explicit formulas for $\Delta^{+(3)}_m$, $H^{\mu(4)}_m$, $\Delta^{+(4)}_m$ and the recursion relation (A.16) imply that

- (in odd dimensions) $\Delta^{+(2l+1)}_m$ is smooth in $m \geq 0$;
- (in even dimensions) $\Delta^{+(2l)}_m$ contains a term which behaves as $m^{2(l-1)}\log(m^2/\mu^2)$ for $m \to 0$;
-

$$H^{\mu\,(4+2k)}_m(y) \equiv \Delta^{+(4+2k)}_m - \pi^{-k}\, m^{2(k+1)}\, f^{(k)}(m^2y^2)\log(m^2/\mu^2) \tag{A.17}$$

(where $f^{(k)}$ is the k-th derivative of f (A.9)) is smooth in $m \geq 0$ and has the same properties as $H_m^{\mu\,(4)}$: its antisymmetric part is $= i\Delta_m^{(d)}/2$ (where $d \equiv 4 + 2k$), it is Poincaré invariant, $\mathrm{WF}(H_m^{\mu\,(d)}) = \mathrm{WF}(\Delta_m^{+(d)})$, it solves the Klein-Gordon equation (since $(\square_y^{(4+2k)} + m^2)\, f^{(k)}(m^2\, y^2) = 0$), it scales almost homogeneously with degree $(d-2)$ and power 1, and $H_{m=0}^{\mu\,(d)} = \Delta_{m=0}^{+(d)}$. Due to the statements about the antisymmetric part, Poincaré invariance and the wave front set, $H_m^{\mu\,(d)}$ can be used for the definition (2.6) of the $*$-product.

B Extension of a distribution to a point

We review in this Appendix the proofs of Theorem 3 and Proposition 4 (given in [7] and [31] respectively) and add some completions. Similar or related techniques can be found in the older works [26, 42] and [18].

In case (a) of Theorem 3 the extension is obtained by the following limit: let χ be a smooth function on $\mathbb{R}^k$ such that $0 \leq \chi \leq 1, \quad \chi(x) = 0$ for $|x| < 1$ and $\chi(x) = 1$ for $|x| > 2$. One can show that the following limit

$$(t, h) \stackrel{\mathrm{def}}{=} \lim_{\rho \to \infty} (t^\circ(x), \chi(\rho x) h(x)) \tag{B.1}$$

(note $\chi(\rho x)h(x) \in \mathcal{D}(\mathbb{R}^k \setminus \{0\})$) exists, and that the so defined t fulfills $\mathrm{sd}(t) = \mathrm{sd}(t^\circ)$.

The construction of the extensions in the most interesting case $k \leq \mathrm{sd}(t^\circ) < \infty$ (part (b) of Theorem 3) proceeds as follows. Let

$$\omega \stackrel{\mathrm{def}}{=} \mathrm{sd}(t^\circ) - k\ , \quad \mathcal{D}_\omega(\mathbb{R}^k) \stackrel{\mathrm{def}}{=} \{h \in \mathcal{D}(\mathbb{R}^k) \mid \partial^a h(0) = 0\, \forall |a| \leq \omega\}. \tag{B.2}$$

We will see that t° has a unique extension t_ω to $\mathcal{D}_\omega \equiv \mathcal{D}_\omega(\mathbb{R}^k)$ and that each projector W from $\mathcal{D} \equiv \mathcal{D}(\mathbb{R}^k)$ onto $\mathcal{D}_\omega$ yields an extension $t \in \mathcal{D}'(\mathbb{R}^k)$ (with $\mathrm{sd}(t) = \mathrm{sd}(t^\circ)$) by $(t, h) \stackrel{\mathrm{def}}{=} (t_\omega, Wh)$ (which is called 'W-extension').

There are many possibilities to construct such a projector W or, equivalently, to choose a corresponding complementary space $\mathcal{E} = \mathrm{ran}(1 - W)$ of $\mathcal{D}_\omega$ in $\mathcal{D}$. (By 'complementary space' we mean: $\mathcal{D} = \mathcal{D}_\omega \oplus \mathcal{E}$). The following Lemma gives a parametrisation of this possibilities in terms of a set of functions:

Lemma 11. *[7]: (a) For any set of functions*

$$\{w_a \in \mathcal{D} \mid a \in \mathbb{N}_0^k, |a| \le \omega, \partial^b w_a(0) = \delta_a^b \, \forall b \in \mathbb{N}_0^k\}, \tag{B.3}$$

the linear map

$$W : \mathcal{D} \longrightarrow \mathcal{D} \, : \, Wh(x) = h(x) - \sum_{|a|\le\omega} (\partial^a h)(0) w_a(x) \tag{B.4}$$

is a projector onto $\mathcal{D}_\omega$.
(b) Conversely, given a projector W from $\mathcal{D}$ onto $\mathcal{D}_\omega$ (or equivalently a complementary space $\mathcal{E}$ of $\mathcal{D}_\omega$ in $\mathcal{D}$), then there exist functions $(w_a)_a$ with the properties (B.3), such that W can be expressed in terms of the $(w_a)_a$ by (B.4).

An example for the functions $(w_a)_{|a|\le\omega}$ is $w_a(x) = \frac{x^a}{a!} w(x)$ where $w \in \mathcal{D}(\mathbb{M})$ and $w|_\mathcal{U} \equiv 1$ for some neighborhood $\mathcal{U}$ of $x = 0$.

Proof. [24] (a) is obvious. To prove (b) we first show that there exists a basis $(w_a)_{|a|\le\omega}$ of the vector space $\mathcal{E} = \mathrm{ran}(1 - W)$ with $\partial^b w_a(0) = \delta_a^b$. The decomposition $\mathcal{D} = \mathcal{D}_\omega \oplus \mathcal{E}$ induces a decomposition of the dual space $\mathcal{D}' = \mathcal{D}'_\omega \oplus \mathcal{D}_\omega^\perp$ by the prescriptions $(f_2, h_1) = 0 \wedge (f_1, h_2) = 0, \forall f_2 \in \mathcal{D}'_\omega, h_1 \in \mathcal{E}, f_1 \in \mathcal{D}_\omega^\perp$, $h_2 \in \mathcal{D}_\omega$. A basis of $\mathcal{D}_\omega^\perp$ is given by $(\partial^a \delta)_{|a|\le\omega}$. We define $((-1)^{|a|} w_a)_{|a|\le\omega}$ to be the dual basis (in $\mathcal{E}$), and it obviously has the properties (B.3).

So, for any $h \in \mathcal{D}$, $(1-W)h$ can be written as $(1-W)h(x) = \sum_a c^a w_a(x)$ with $c_a \in \mathbb{C}$, and we find $\partial^b h(0) = \partial^b (1 - W)h(0) = c^b$, $|b| \le \omega$. Hence, $Wh(x) = h(x) - \sum_a (\partial^a h)(0) w_a(x)$. □

We split any $h \in \mathcal{D}$ into $h = h_1 + h_2$, $h_1 = \sum_{|a|\le\omega} (\partial^a h)(0) w_a \in \mathcal{E}, h_2 \in \mathcal{D}_\omega$. h_2 has the form

$$Wh(x) \equiv h_2(x) = \sum_{|a|=[\omega]+1} x^a g_a(x), \quad \text{with} \quad g_a \in \mathcal{D}. \tag{B.5}$$

This decomposition of h_2 is non-unique in general, however, we will see that this does not matter. From (3.6) we recall $\mathrm{sd}(x^b f) \le \mathrm{sd}(f) - |b|, \forall f \in \mathcal{D}'(\mathbb{R}^k)$ or $\mathcal{D}'(\mathbb{R}^k \setminus \{0\})$. For $|a| = [\omega] + 1$ we find $\mathrm{sd}(x^a t^\circ) \le \omega + k - ([\omega] + 1) < k$. Therefore, from part (a) of Theorem 3 we know that $x^a t^\circ \in \mathcal{D}'(\mathbb{R}^k \setminus \{0\})$ has a unique extension $\overline{x^a t^\circ} \in \mathcal{D}'(\mathbb{R}^k)$. Now we define $t \in \mathcal{D}'(\mathbb{R}^k)$ by

$$(t, h) \stackrel{\text{def}}{=} \sum_{|a|=[\omega]+1} (\overline{x^a t^\circ}, g_a) + \sum_{|a|\le\omega} C_a (\partial^a h)(0), \quad h \in \mathcal{D}(\mathbb{R}^k), \tag{B.6}$$

[24]The idea of proof is given in [7].

where the $C_a \in \mathbb{C}$ are arbitrary constants. By means of (B.1) we find for the first term

$$(t, Wh) = \sum_{|a|=[\omega]+1} (\overline{x^a t^\circ}, g_a) = \lim_{\rho\to\infty} (t^\circ(x), \chi(\rho x) Wh(x)). \qquad \text{(B.7)}$$

Hence, $\sum_{|a|=[\omega]+1}(\overline{x^a t^\circ}, g_a)$ is independent of the choice of the decomposition (B.5). Obviously, t is an extension of t°, and in [7] it is proved $\mathrm{sd}(t) = \mathrm{sd}(t^\circ)$. $t|_{\mathcal{D}_\omega} \equiv t_\omega$ is uniquely fixed (B.7), but $t|_{\mathcal{E}}$ is completely arbitrary, namely $(t, w_a) = C_a$ can be arbitrarily chosen. This is the freedom of normalization in perturbative renormalization (3.8), which gives rise to the renormalization group (see Sect. 4).

In case (c) of Theorem 3 there exists a linear functional $\bar{t}$ on $\mathcal{D}(\mathbb{R}^k)$ which fulfills $\bar{t}(h) = (t^\circ, h) \quad \forall h \in \mathcal{D}(\mathbb{R}^k \setminus \{0\})$, according to the Hahn-Banach theorem. But $\bar{t}$ is not continuous, i.e. it is not a distribution.

It is useful to know that given an extension t of t°, there exists a projector W from $\mathcal{D}$ onto $\mathcal{D}_\omega$ such that $t = t \circ W$ (i.e. all constants C_a in (B.6) vanish), in detail:

Lemma 12. *Let* $\omega \stackrel{\text{def}}{=} \mathrm{sd}(t^\circ) - k$ *and let an extension* t *of* t° *be given with* $\mathrm{sd}(t) = \mathrm{sd}(t^\circ)$.
(a) Then there exists a complementary space $\mathcal{E}$ *of* $\mathcal{D}_\omega$ *in* $\mathcal{D}$ *with* $t|_{\mathcal{E}} = 0$.
(b) There exist functions $w_a \in \mathcal{D}$, $a \in \mathbb{N}_0^k, |a| \leq \omega$ *with* $\partial^b w_a(0) = \delta_a^b$ *and* $t = t \circ W$, *where* W *is given in terms of the* $(w_a)_a$ *by (B.4).*

Proof. By part (b) of Lemma 11, the statement (b) is a consequence of (a). To prove (a) let $\mathcal{E}_1$ be a complementary space of $\mathcal{D}_\omega$ in $\mathcal{D}$ and W_1 the corresponding projector on $\mathcal{D}_\omega$. We choose a $g \in \mathcal{D}_\omega$ with $(t, g) = 1$. Now we set

$$\mathcal{E} \stackrel{\text{def}}{=} \{k - (t, k)g \mid k \in \mathcal{E}_1\}. \qquad \text{(B.8)}$$

Obviously $\mathcal{E}$ is a vector space and it holds $t|_{\mathcal{E}} = 0$. To see $\mathcal{D} = \mathcal{D}_\omega + \mathcal{E}$ we decompose any $h \in \mathcal{D}$ into $h = h_1 + h_2$, $h_1 \in \mathcal{E}_1, h_2 \in \mathcal{D}_\omega$. Then, $h = (h_1 - (t, h_1)g) + (h_2 + (t, h_1)g)$, $(h_1 - (t, h_1)g) \in \mathcal{E}$, $(h_2 + (t, h_1)g) \in \mathcal{D}_\omega$. It remains to show $\mathcal{E} \cap \mathcal{D}_\omega = \{0\}$. Let $l \in \mathcal{E} \cap \mathcal{D}_\omega$. So, $l = k - (t, k)g$ for some $k \in \mathcal{E}_1$, and on the other hand $l = W_1 l = W_1 k - (t, k)W_1 g = -(t, k)g$. We find $k = 0$ and hence $l = 0$. □

Obviously the W-extension (B.7) is a *non-local* renormalization prescription: it depends on $t^\circ|_{\mathcal{D}(U)}$ where $U := \cup_{|a|\leq\omega} \mathrm{supp}\, w_a$. In contrast the condition of almost homogeneous scaling ensures that the extension depends on

the short distance behavior of t° only. We now prove that the latter condition can be maintained, following to a large extent [31].

Proof of Proposition 4. Let t_1 be any extension of t° with $\mathrm{sd}(t_1) = \mathrm{sd}(t^\circ) = D$. Since t° scales almost homogeneously (with power N) the support of $(x\partial_x + D)^{N+1} t_1(x)$ must be contained in $\{0\}$. In addition it holds $\mathrm{sd}((x\partial_x + D)^{N+1} t_1) = \mathrm{sd}(t_1) = D$, and hence

$$(\sum_r x_r \partial_{x_r} + D)^{N+1} t_1(x) = \sum_{|a| \le D-k} C_a \partial^a \delta^{(k)}(x) \ . \tag{B.9}$$

We will frequently use

$$(\sum_r x_r \partial_{x_r} + D)\partial^a \delta^{(k)}(x) = (D - k - |a|)\partial^a \delta^{(k)}(x) \ . \tag{B.10}$$

- For $D \notin \mathbb{N}_0 + k$ we may set

  $$t \overset{\mathrm{def}}{=} t_1 - \sum_{|a| \le D-k} \frac{C_a}{(D - k - |a|)^{N+1}} \partial^a \delta^{(k)} \ , \tag{B.11}$$

  This is an extension which maintains even the power of the almost homogeneous scaling.

- If $D \in \mathbb{N}_0 + k$ the subtraction of the $\partial^a\delta$-terms in (B.11) works not for $|a| = D - k$. We can only perform the finite renormalization

  $$t \overset{\mathrm{def}}{=} t_1 - \sum_{|a| < D-k} \frac{C_a}{(D - k - |a|)^{N+1}} \partial^a \delta^{(k)} \ . \tag{B.12}$$

  With that

  $$(\sum_r x_r \partial_{x_r} + D)^{N+1} t = \sum_{|a| = D-k} C_a \partial^a \delta^{(k)} \ . \tag{B.13}$$

  However, applying the operator $(x\partial_x + D)$ once more we get zero, i.e. t scales almost homogeneously with power $\le N + 1$.

The statements about the uniqueness of t are obvious, because $\partial^a \delta^{(k)}$ scales homogeneously with degree $(k + |a|)$. □

How to find an extension t of t° (with $\mathrm{sd}(t) = \mathrm{sd}(t^\circ)$) in practice? For $\mathrm{sd}(t^\circ) < k$ this is trivial: t is given by the same formula, the domain may be extended by continuity (B.1). But for $k \leq \mathrm{sd}(t^\circ) < \infty$ the map W (B.4) gets complicated in explicit calculations. (An exception are purely massive theories in which one may choose $w_a = \frac{x^a}{a!} \notin \mathcal{D}$ in (B.3)-(B.4); this gives the 'central solution' of Epstein and Glaser [18].) A construction of $R_{1,1}$ (or equivalently T_2) is given in Appendix B. It uses the Källen-Lehmann representation of the commutator of two Wick polynomials, and hence, it is unclear how to generalize this method to higher orders. It seems that **differential renormalization** [21, 35, 39] is a practicable way to trace back the case $k \leq \mathrm{sd}(t^\circ) < \infty$ to the trivial case $\mathrm{sd}(t^\circ) < k$ in arbitrary high orders. The idea is to write t° as a derivative of a distribution $f^\circ \in \mathcal{D}'(\mathbb{R}^k \setminus \{0\})$ with $\mathrm{sd}(f^\circ) < k$; more precisely

$$t^\circ = Df^\circ \qquad \text{where} \qquad D = \sum_{|a|=l} C_a \partial^a \quad (C_a \in \mathbb{C}) \tag{B.14}$$

such that $\mathrm{sd}(f^\circ) = \mathrm{sd}(t^\circ) - l < k$. Let $f \in \mathcal{D}'(\mathbb{R}^k)$ be the unique extension of f° with $\mathrm{sd}(f) = \mathrm{sd}(f^\circ)$. Then,

$$t \stackrel{\mathrm{def}}{=} Df \tag{B.15}$$

solves the extension problem with $\mathrm{sd}(t) = \mathrm{sd}(t^\circ)$. The non-uniqueness of t shows up in the non-uniqueness of f°: one may add to f° a distribution $g^\circ \in \mathcal{D}'(\mathbb{R}^k \setminus \{0\})$ with $\mathrm{sd}(g^\circ) = \mathrm{sd}(f^\circ)$ and $Dg^\circ = 0$ on $\mathbb{R}^k \setminus \{0\}$. The (unique) extension g of g° (with $\mathrm{sd}(g) = \mathrm{sd}(g^\circ)$) fulfills $\mathrm{supp}\, Dg \subset \{0\}$ and, hence, the addition of g° can change t (B.15) only by a local term. For example let $\mathrm{sd}(t^\circ) = k$ and $D = \square$. Then, g is of the form: $g = \alpha D^{\mathrm{ret}} +$ (solution of the homogeneous differential equation) ($\alpha \in \mathbb{C}$), and this yields $t_{\mathrm{new}} \equiv D(f+g) = t_{\mathrm{old}} + \alpha\, \delta$.

Example: **differential renormalization of the massless fish and setting-sun diagram,**

$$\begin{aligned} r_0^\circ(y) &= j_0(y)\, \Theta(-y^0)\ , \quad j_0(y) \equiv \Big(\frac{1}{(y^2 - iy^0 0)^2} - \frac{1}{(y^2 + iy^0 0)^2} \Big)\ , \\ r_1^\circ(y) &= j_1(y)\, \Theta(-y^0)\ , \quad j_1(y) \equiv \Big(\frac{1}{(y^2 - iy^0 0)^3} - \frac{1}{(y^2 + iy^0 0)^3} \Big)\ , \end{aligned} \tag{B.16}$$

and of

$$r_2^\circ(y) = j_2(y)\,\Theta(-y^0)\ , \quad j_2(y) \equiv \Big(\frac{\log(-\mu^2(y^2-iy^00))}{(y^2-iy^00)^2} - (y\to -y)\Big) \tag{B.17}$$

for $d=4$, cf. Appendix C. These r°-distributions appear in the Example (3.19)-(3.22).[25] In agreement with Lemma 1(b) the j_l's have support in $\bar V_+ \cup \bar V_-$. We are looking for distributions J_l with

$$\mathrm{sd}(J_l) < 4\ , \quad \mathrm{supp}\, J_l \subset (\bar V_+ \cup \bar V_-) \quad \text{and} \quad j_l = D_l\, J_l \tag{B.18}$$

where D_l is a power of the wave operator. Due to the lowered scaling degree of J_l, the product $J_l(y)\,\Theta(-y^0)$ exits in $\mathcal{D}'(\mathbb{R}^4)$ and one easily verifies that

$$r_l(y) \stackrel{\mathrm{def}}{=} D_l\,\big(J_l(y)\,\Theta(-y^0)\big) \tag{B.19}$$

is a Lorentz invariant extension of r_l° with the same scaling degree. With some trial and error one finds

$$\begin{aligned} j_0(y) &= \Box_y \Big(\frac{-\log(-\mu^2(y^2-iy^00))}{4\,(y^2-iy^00)} - (y\to -y)\Big)\ ,\\ j_1(y) &= \Box_y\Box_y \Big(\frac{-\log(-\mu^2(y^2-iy^00))}{32\,(y^2-iy^00)} - (y\to -y)\Big)\ ,\\ j_2(y) &= \Box_y \Big(\frac{-(\log(-\mu^2(y^2-iy^00)))^2 - 2\log(-\mu^2(y^2-iy^00))}{8\,(y^2-iy^00)} - (y\to -y)\Big)\ . \end{aligned} \tag{B.20}$$

In case of the fish and the setting sun diagram a scale $\mu > 0$ is introduced; this cannot be avoided by using other methods of renormalization either, cf. Appendix C. If we would replace $(-\mu^2)$ by μ^2 in J_0 and J_1 the relation $j_l = D_l\, J_l$ would still hold, but J_0 and J_1 would have support in $\{y|y^2 \le 0\}$. This alternative possibility to fulfill $j_l = D_l\, J_l$ reflects the peculiarity that j_0 and j_1 vanish on $\mathcal{D}(\{y|y^2>0\})$. All J_l's scale almost homogeneously with degree 2 and the corresponding power is the power of $\log(\mu^2...)$. We explicitly see that in all three examples the extension increases this power by 1. For the breaking of homogeneous scaling of the fish and setting sun diagram we

[25] We omit constant pre-factors.

obtain

$$\rho^4\, r_0(\rho y) - r_0(y) = i\pi \log\rho\ \square_y\Big(\Theta(-y^0)\,\delta(y^2)\Big) = i2\pi^2 \log\rho\ \ \delta(y)\ , \quad \text{(B.21)}$$

$$\rho^6\, r_1(\rho y) - r_1(y) = \frac{i\pi}{8}\log\rho\ \square_y\square_y\Big(\Theta(-y^0)\,\delta(y^2)\Big) = \frac{i\pi^2}{4}\log\rho\ \square\delta(y)\ , \quad \text{(B.22)}$$

where we use $\Theta(-y^0)\,\delta(y^2) \sim D^{\mathrm{ret}}(-y)$ and $\square D^{\mathrm{ret}} = \delta$. In Appendix C these results are obtained by means of another method of renormalization, which is more straightforward.

Remark: The construction in the proof of Theorem 8 yields also an extension t of the given t° if one works in (B.2), (B.5) and (B.6) with an ω which is strictly greater than $(\mathrm{sd}(t^\circ) - k)$; but then it holds genericly $\mathrm{sd}(t) = \omega + k > \mathrm{sd}(t^\circ)$. (This is called an 'over-subtracted' extension.)

C Extension of two-point functions

In this Appendix the number of space time dimensions is $d = 4$. The x-space method which we give here to renormalize the fish diagram

$$r_0(y) \stackrel{\mathrm{def}}{=} r_{1,1}(\varphi^2, \varphi^2)(y)\ , \qquad y \equiv x_1 - x\ , \quad \text{(C.1)}$$

and the setting-sun diagram

$$r_1(y) \stackrel{\mathrm{def}}{=} r_{1,1}(\varphi^3, \varphi^3)(y)\ , \quad \text{(C.2)}$$

can be used for arbitrary first order terms $r_{1,1}(A_1, A_2)$, $A_1, A_2 \in \mathcal{P}$. (See footnote 12 for the notation.) We treat here the massless case, which needs additional care to avoid IR-divergences. We first compute the fish diagram by following our inductive construction of Sect. 3. A straightforward calculation yields

$$\begin{aligned} j_0(y) &\stackrel{\mathrm{def}}{=} \omega_0\Big([\varphi^2(x_1), \varphi^2(x)]_\star\Big) \\ &= 2\big(D^+(y)^2 - D^+(-y)^2\big) = \frac{i}{2(2\pi)^2}\int_0^\infty dm^2\, \Delta_m(y)\ , \end{aligned} \quad \text{(C.3)}$$

which is the Källen-Lehmann representation. We recall $\Delta_m(y) = \Delta_m^{\mathrm{ret}}(y) - \Delta_m^{\mathrm{ret}}(-y)$ with $(\square + m^2)\Delta_m^{\mathrm{ret}} = \delta$ and $\mathrm{supp}\,\Delta_m^{\mathrm{ret}} \subset \bar{V}_+$. According to the

axioms Causality, GLZ relation and Scaling the wanted distribution r_0 is determined by

$$\mathrm{supp}\, r_0 \subset \bar{V}_- \ , \quad (r_0, h) = -i(j_0, h) \quad \forall h \in \mathcal{D}(\bar{V}_- \setminus \{0\}) \tag{C.4}$$

and

$$(\rho\partial_\rho + 4)^2 r_0(\rho y) = 0 \ . \tag{C.5}$$

If we replace $\Delta_m(y)$ in (C.3) by $-\Delta_m^{\mathrm{ret}}(-y)$ the m^2-integral becomes UV-divergent. However, for $\mu > 0$ one easily verifies that

$$r_{0\,\mu}(y) \stackrel{\mathrm{def}}{=} \frac{1}{2(2\pi)^2}(\Box_y - \mu^2) \int_0^\infty dm^2 \, \frac{\Delta_m^{\mathrm{ret}}(-y)}{m^2+\mu^2} \tag{C.6}$$

exists as a distribution and solves (C.4). To avoid an IR-divergence, we need to introduce a scale $\mu > 0$, which breaks homogeneous scaling (because of $\rho^4 r_{0\,\mu}(\rho y) = r_{0\,\rho\mu}(y)$). To verify the scaling requirement (C.5) we compute

$$\begin{aligned}
\rho^4 r_{0\,\mu}(\rho y) - r_{0\,\mu}(y) &= r_{0\,\rho\mu}(y) - r_{0\,\mu}(y) \\
= \frac{1}{2(2\pi)^2}\Big(((\Box_y - \mu^2) &+ (1-\rho^2)\mu^2) \int dm^2 \, \frac{\Delta_m^{\mathrm{ret}}(-y)}{m^2+\rho^2\mu^2} \\
&-(\Box_y - \mu^2) \int dm^2 \, \frac{\Delta_m^{\mathrm{ret}}(-y)}{m^2+\mu^2}\Big) \\
= \frac{1}{2(2\pi)^2}\Big[\int dm^2 \, (\Box_y - \mu^2)&\Delta_m^{\mathrm{ret}}(-y)\Big(\frac{1}{m^2+\rho^2\mu^2} - \frac{1}{m^2+\mu^2}\Big) \\
+(1-\rho^2)\mu^2 \int dm^2 \, \frac{\Delta_m^{\mathrm{ret}}(-y)}{m^2+\rho^2\mu^2}\Big] &= \frac{1}{(2\pi)^2} \log\rho \;\; \delta(y) \ .
\end{aligned} \tag{C.7}$$

Hence, (C.5) is indeed satisfied for all $\mu > 0$. So we get explicitly the breaking of homogeneous scaling without really computing the integral (C.6). As a byproduct the calculation (C.7) shows explicitly that the choice of $\mu > 0$ is precisely the choice of the indeterminate parameter C in the general solution $r_0(y) + C\delta(y)$.[26]

[26]Note that

$$\begin{aligned}
r_{0\,\mu\nu}(y) &\equiv \frac{-1}{2(2\pi)^2}(-\Box_y+\mu^2)(-\Box_y+\nu^2)\int_0^\infty dm^2 \, \frac{\Delta_m^{\mathrm{ret}}(-y)}{(m^2+\mu^2)(m^2+\nu^2)} \\
&= r_{0\,\mu}(y) + \frac{1}{2(2\pi)^2}\int_0^\infty dm^2 \, \frac{1}{(m^2+\mu^2)(m^2+\nu^2)} \cdot (\Box - \mu^2)\delta(y)
\end{aligned} \tag{C.8}$$

solves also (C.4), but it violates (C.5): the term $\sim \Box\delta(y)$ scales homogeneously with degree 6 (instead of 4).

We proceed analogously for the setting-sun diagram:

$$\begin{aligned} j_1(y) &\stackrel{\text{def}}{=} \omega_0\Big([\varphi^3(x_1), \varphi^3(x)]_\star\Big) \\ &= 6\big(D^+(y)^3 - D^+(-y)^3\big) = \frac{3i}{16(2\pi)^4}\int_0^\infty dm^2\, m^2\, \Delta_m(y)\ . \end{aligned} \tag{C.9}$$

Obviously,

$$r_{1\,\mu_1,\mu_2}(y) = \frac{-3}{16(2\pi)^4}(\square_y - \mu_1^2)(\square_y - \mu_2^2)\int_0^\infty dm^2\, \frac{m^2\,\Delta_m^{\text{ret}}(-y)}{(m^2+\mu_1^2)(m^2+\mu_2^2)} \tag{C.10}$$

(where $\mu_1, \mu_2 > 0$) solves (C.4). But we are looking for solutions r_1 of (C.4) which additionally scale almost homogeneously with degree 6 and power ≤ 1, i.e. $(\rho\partial_\rho + 6)^2 r_1(\rho y) = 0$. For $r_{1\,\mu,\mu}$ the breaking of homogeneous scaling is equal to

$$\begin{aligned} \rho^6 r_{1\,\mu,\mu}(\rho y) - r_{1\,\mu,\mu}(y) &= (r_{1\,\rho\mu,\rho\mu}(y) - r_{1\,\mu,\rho\mu}(y)) + (r_{1\,\mu,\rho\mu}(y) - r_{1\,\mu,\mu}(y)) \\ &= \frac{-3}{16(2\pi)^4}\Big[-\square_y\delta(y)\,(\log\rho^2) + \delta(y)\,\mu^2\,(\rho^2-1)\Big]\ , \end{aligned} \tag{C.11}$$

where the method (C.7) is used twice. We see that $r_{1\,\mu,\mu}$ violates our scaling condition and, by a generalization of the calculation (C.11), one finds that this holds true even for all $r_{1\,\mu_1,\mu_2}$, $(\mu_1,\mu_2) \in \mathbb{R}^+ \times \mathbb{R}^+$. However, from the result (C.11) we read off that

$$r_1(y) \stackrel{\text{def}}{=} r_{1\,\mu,\mu}(y) + \frac{3}{16(2\pi)^4}\,\mu^2\,\delta(y) + C_2\square\delta(y) \tag{C.12}$$

(where $C_2 \in \mathbb{R}$ is arbitrary) fulfills our requirements. (C.12) is the most general solution which is additionally Lorentz invariant and unitary. For the breaking of homogeneous scaling we obtain

$$\rho^6 r_1(\rho y) - r_1(y) = \frac{3}{8(2\pi)^4}\,(\log\rho)\ \square\delta(y)\ . \tag{C.13}$$

A general fact shows up in the results (C.7) and (C.13) (which is also valid for $m > 0$): the breaking of homogeneous scaling is independent of the normalization (i.e. of the choice of μ in (C.6) and C_2 in (C.12)), because the undetermined polynomial $\sum_{|a|+l=\omega} C_{a,l} m^l \partial^a \delta(y)$ scales homogeneously.

D Maintenance of symmetries in the extension of distributions

In contrast to a large part of the literature we work in this Appendix with our normalization conditions Smoothness in $m \geq 0$ and Scaling, instead of the upper bound (3.7) on the scaling degree. However, by obvious modifications, the procedure given here can just as well be based on the latter normalization condition. We investigate the question whether symmetries can be maintained in the process of renormalization. Or in mathematical terms: given a $t^\circ \equiv t^{(m)\circ} \in \mathcal{D}'(\mathbb{R}^k \setminus \{0\})$ which is smooth in $m \geq 0$ and scales almost homogeneously with degree D and power N (3.9), does there exist an extension $t \equiv t^{(m)} \in \mathcal{D}'(\mathbb{R}^k)$ with the same symmetries and smoothness (in m) as t° and which scales almost homogeneously with D and power $\leq (N+1)$?

Existence of a symmetric extension: let V be a representation of a group G on $\mathcal{D}(\mathbb{R}^k)$ under which $\mathcal{D}(\mathbb{R}^k \setminus \{0\})$ and t° are invariant,

$$(t^\circ, V(g)h) = (t^\circ, h) \qquad \forall h \in \mathcal{D}(\mathbb{R}^k \setminus \{0\}) \ , \quad g \in G \ . \tag{D.1}$$

We denote by V^T the transposed representation of G on $\mathcal{D}'(\mathbb{R}^k)$

$$(V^T(g)s, h) = (s, V(g^{-1})h) \ , \qquad s \in \mathcal{D}'(\mathbb{R}^k) \ , \tag{D.2}$$

and we additionally assume that smoothness in $m \geq 0$ and the scaling behavior (3.9) are maintained under $V^T(g)$, $\forall g \in G$.

Let $t \in \mathcal{D}'(\mathbb{R}^k)$ be an arbitrary extension of t° with the required smoothness and scaling properties. For $h \in \mathcal{D}(\mathbb{R}^k \setminus \{0\})$ we know

$$(V^T(g)t, h) = (t, V(g^{-1})h) = (t^\circ, V(g^{-1})h) = (V^T(g)t^\circ, h) \ . \tag{D.3}$$

So, $V^T(g)t$ is an extension of $V^T(g)t^\circ = t^\circ$. With (3.16) we conclude[27]

$$l(g) \overset{\text{def}}{=} V^T(g)t - t \in \mathcal{D}^\perp_\omega(\mathbb{R}^k) \overset{\text{def}}{=} \Big\{ \sum_{|a|+l=\omega} m^l C_{l,a} \partial^a \delta^{(k)} \mid C_{l,a} \in \mathbb{R} \text{ or } \mathbb{C} \Big\} \tag{D.4}$$

where $\omega \overset{\text{def}}{=} D - k$. For an $\tilde{l} \in \mathcal{D}^\perp_\omega(\mathbb{R}^k)$ we find $V^T(g)\tilde{l} = \big(V^T(g)(t+\tilde{l}) - V^T(g)t\big) \in \mathcal{D}^\perp_\omega(\mathbb{R}^k)$, since both terms are an extension of t°. Hence,

$$G \ni g \longrightarrow \pi(g) \overset{\text{def}}{=} V^T(g)|_{\mathcal{D}^\perp_\omega(\mathbb{R}^k)} \tag{D.5}$$

[27] The case $m = 0$ is included by using $m^l|_{m=0} = \delta_{l,0}$.

is a sub-representation of V^T.

We are searching an $l_0 \in \mathcal{D}^{\perp}_{\omega}(\mathbb{R}^k)$ such that $t + l_0$ is invariant,

$$V^T(g)(t + l_0) = t + l(g) + V^T(g)l_0 = t + l_0 \ , \tag{D.6}$$

i.e. l_0 must fulfill

$$l(g) = l_0 - \pi(g)l_0 \ , \qquad \forall g \in G \ . \tag{D.7}$$

From (D.4) it follows that $l(g)$ has the property

$$l(gh) = V^T(gh)t - t = V^T(g)\left(V^T(h)t - t\right) + V^T(g)t - t = \pi(g)l(h) + l(g) \ . \tag{D.8}$$

A solution of such an equation is called a 'cocycle'. If $l(g)$ is of the form (D.7) for some l_0, it is called a 'coboundary', and such an $l(g)$ solves automatically the cocycle equation (D.8). The space of the cocycles modulo the coboundaries is called the cohomology of the group with respect to the representation π. Summing up, *the invariance of t° with respect to the representation V^T of G can be maintained in the extension if the cohomology of G with respect to π (D.5) is trivial.*

We are now going to show that *this supposition holds true if all finite dimensional representations of G are completely reducible.* For this purpose we consider the restriction of $V^T(g)$ to the space $(\mathbb{C} \cdot t) \oplus \mathcal{D}^{\perp}_{\omega}(\mathbb{R}^k)$. From (D.4) we see that this is a finite dimensional representation of G, which may be identified with the matrix representation

$$g \longrightarrow \overline{\pi}(g) = \begin{pmatrix} 1 & 0 \\ l(g) & \pi(g) \end{pmatrix} \ . \tag{D.9}$$

Due to the complete reducibility of $\overline{\pi}$, there exists a 1-dimensional invariant subspace U which is complementary to the representation space of π (D.5). Such a subspace is of the form $U = \mathbb{C} \cdot \begin{pmatrix} 1 \\ l_0 \end{pmatrix}$. So there exists an $l_0 \in \mathcal{D}^{\perp}_{\omega}(\mathbb{R}^k)$ with

$$\overline{\pi}(g) \begin{pmatrix} 1 \\ l_0 \end{pmatrix} = \begin{pmatrix} 1 \\ l(g) + \pi(g)l_0 \end{pmatrix} \in \mathbb{C} \cdot \begin{pmatrix} 1 \\ l_0 \end{pmatrix} \ . \tag{D.10}$$

Hence, $\overline{\pi}|_U = \mathbf{1}$, which means that l_0 solves (D.7). □

For the Lorentz group $\mathcal{L}^{\uparrow}_{+}$ all finite dimensional representations are completely reducible and, hence, Lorentz invariance can be maintained in perturbative renormalization.

However, for the scaling transformations one has to consider the representations of $\mathbb{R}_+$ as a multiplicative group. They are not always completely reducible. An example for a reducible but not completely reducible representation is

$$\mathbb{R}_+ \ni \rho \mapsto \begin{pmatrix} 1 & 0 \\ \ln \rho & 1 \end{pmatrix} . \tag{D.11}$$

The existence of such representations can be understood as the reason for the breaking of homogeneous scaling.

Example: The action of massless QED is invariant with respect to the following $U(1)$ transformations

$$U(1)_V \, : \, \psi(x) \to e^{i\alpha(x)}\psi(x) \, , \qquad U(1)_A \, : \, \psi(x) \to e^{i\beta(x)\gamma_5}\psi(x) \, , \tag{D.12}$$

$\alpha(x), \beta(x) \in [0, 2\pi)$, and $\bar{\psi}$ is transformed correspondingly. According to Noether's Theorem the corresponding currents

$$j^\mu_{V\, g\mathcal{L}} = -(\bar{\psi}\gamma^\mu\psi)_{g\mathcal{L}} \quad \text{and} \quad j^\mu_{A\, g\mathcal{L}} = -(\bar{\psi}\gamma^\mu\gamma^5\psi)_{g\mathcal{L}} \tag{D.13}$$

are conserved in classical field theory. In QFT the just derived result implies that invariance of the retarded products with respect to $U(1)_V \times U(1)_A$ transformations (D.12) can be realized, because this group is compact. But it is well known that conservation of $j^\mu_{A\,\mathcal{L}}$ is not compatible with conservation of $j^\mu_{V\,\mathcal{L}}$, this is the axial anomaly. So, in general Noether's Theorem cannot be fulfilled in QFT, see however [9].

Construction of a Lorentz-invariant extension: there remains the question how to find the solution l_0 of the equation (D.7) (if it exists). For compact groups this can be done in the following way: we set

$$l_0 \stackrel{\text{def}}{=} \int_G \mathrm{d}g \, l(g) \, , \tag{D.14}$$

where $\mathrm{d}g$ is the uniquely determined measure on G which has norm 1 and is invariant under left- and right-translations (Haar-measure). To verify that (D.14) solves (D.7) we use the cocycle equation:

$$l_0 - \pi(g)l_0 = \int_G \mathrm{d}h \, (l(h) - \pi(g)l(h)) = \int_G \mathrm{d}h \, (l(h) - l(gh) + l(g)) \ . \tag{D.15}$$

Due to the translation invariance of the Haar-measure, the integrals over the first two terms cancel, and since the measure is normalized we indeed obtain (D.7).

For the Lorentz-group this method fails. Epstein and Glaser [18] use that the renormalized distributions are boundary values of functions which are analytic in a certain region of the complexified Minkowski space and which are additionally invariant under the complex Lorentz group. It suffices then to take into account the invariance with respect to the compact subgroup SO(4).

We give here an alternative method which has some resemblance with [42] and [4]: the strategy is to construct the projector P on the space $\mathcal{D}_{\rm inv}$ of all invariant vectors in the representation $\overline{\pi}$ (D.9). Then, with t being the above used arbitrary extension of t° (with the mentioned smoothness and scaling properties), the definition

$$t_{\rm inv} \stackrel{\rm def}{=} Pt \ , \tag{D.16}$$

yields a $\mathcal{L}_+^\uparrow$-invariant distribution. And it is also an extension of t° with the required properties, because $(t_{\rm inv} - t) \in \mathcal{D}_\omega^\perp(\mathbb{R}^k)$. The latter is shown below.

To find P note that for a Casimir operator C (in any representation) the invariant vectors $\mathcal{D}_{\rm inv}$ are a subspace of its kernel $C^{-1}(0)$, because C is built from the elements of the Lie algebra. Below we will find a Casimir operator C_0 of the Lorentz group with $\mathcal{D}_{\rm inv} = C_0^{-1}(0)$ in each finite dimensional representation. With that our method relies on the fact that the operator $c^{-1}(c\mathbf{1} - C_0)$ annihilates the eigenvectors (of C_0) to the eigenvalue $c \neq 0$, and on $C_0^{-1}(0)$ it is $= \mathbf{1}$. Therefore, in each finite dimensional representation of $\mathcal{L}_+^\uparrow$ and in particular for $\overline{\pi}$ (D.9), the operator

$$P \stackrel{\rm def}{=} \prod_{c\neq 0} \frac{c\mathbf{1} - \overline{\pi}(C_0)}{c} \tag{D.17}$$

(c runs through all eigenvalues $\neq 0$ of C_0) is a projector on $\mathcal{D}_{\rm inv}$.

To show $(t_{\rm inv} - t) \in \mathcal{D}_\omega^\perp(\mathbb{R}^k)$ we write

$$t_{\rm inv} = Pt = P(t + l_0) - Pl_0 = t + l_0 - Pl_0 \ , \tag{D.18}$$

where we use that $t + l_0$ is invariant (D.6). It follows from (D.9) and (D.17) that the upper right coefficient of the matrix P is $= 0$. Hence, $Pl_0 \in \mathcal{D}_\omega^\perp(\mathbb{R}^k)$, and this gives the assertion.

To determine a Casimir operator C_0 of the Lorentz group with $C_0^{-1}(0) = \mathcal{D}_{\rm inv}$ in each finite dimensional representation, we first note that there are two quadratic Casimirs,

$$C_0 = \vec{L}^2 - \vec{M}^2 \quad \text{and} \quad C_1 = \vec{L} \cdot \vec{M} \ , \tag{D.19}$$

where $\vec{L}$ denotes the infinitesimal rotations and $\vec{M}$ the infinitesimal Lorentz-boosts. On $\mathcal{D}(\mathbb{R}^4)$ it holds

$$\vec{L} = \frac{1}{i}\vec{x} \times \vec{\partial} \ , \qquad \vec{M} = \frac{1}{i}(x^0\vec{\partial} - \vec{x}\partial_0) \ . \tag{D.20}$$

The irreducible finite dimensional representations of the Lorentz group $\mathcal{L}_+^\uparrow$ are those irreducible finite dimensional representations of $\mathrm{SL}(2,\mathbb{C})$ which represent the matrix -1 by 1. The irreducible finite dimensional representations of $\mathrm{SL}(2,\mathbb{C})$ are indexed by two spin quantum numbers $j_1, j_2 \in \frac{1}{2}\mathbb{N}_0$ and have the form

$$\pi_{j_1 j_2}(A)\big(\xi^{\otimes 2j_1} \otimes \eta^{\otimes 2j_2}\big) = (A\xi)^{\otimes 2j_1} \otimes ((A^*)^{-1}\eta)^{\otimes 2j_2} \tag{D.21}$$

with $\xi, \eta \in \mathbb{C}^2$. For $j_1 + j_2 \in \mathbb{N}_0$ this yields a representation of the Lorentz group. We denote by $\vec{L}_i, \vec{M}_i$ the representations of $\vec{L}$ and $\vec{M}$ on the left ($i = 1$) and the right factor ($i = 2$) respectively. In the fundamental representation of $\mathrm{SL}(2,\mathbb{C})$ (i.e. $(j_1, j_2) = (\frac{1}{2}, 0)$) we have

$$\vec{L} = \frac{1}{2}\vec{\sigma} \ , \qquad \vec{M} = \frac{i}{2}\vec{\sigma} \ , \tag{D.22}$$

and in the conjugated representation (i.e. $(j_1, j_2) = (0, \frac{1}{2})$) it holds

$$\vec{L} = \frac{1}{2}\vec{\sigma} \ , \qquad \vec{M} = -\frac{i}{2}\vec{\sigma} \ . \tag{D.23}$$

So we have

$$\vec{M}_1 = i\vec{L}_1 \ , \qquad \vec{M}_2 = -i\vec{L}_2 \ , \tag{D.24}$$

and the validity of these relations goes over to all representations π_{j_1,j_2}, $j_1, j_2 \in \frac{1}{2}\mathbb{N}_0$ (D.21). With that we obtain for the Casimir operators

$$C_0 = (\vec{L}_1+\vec{L}_2)^2-(\vec{M}_1+\vec{M}_2)^2 = 2\vec{L}_1^2+2\vec{L}_2^2 = 2(j_1(j_1+1)+j_2(j_2+1)) \tag{D.25}$$

and

$$C_1 = (\vec{L}_1 + \vec{L}_2)\cdot(\vec{M}_1 + \vec{M}_2) = i\vec{L}_1^2 - i\vec{L}_2^2 = i(j_1(j_1+1) - j_2(j_2+1)) \ . \tag{D.26}$$

We find indeed that C_0 vanishes on the trivial representation $j_1 = j_2 = 0$ only. However, the kernel of C_1 is much bigger.

As an example let us consider a Lorentz invariant distribution t_ω on $\mathcal{D}_\omega(\mathbb{R}^4)$, $\omega = 2$. Let

$$W = 1 - \sum_{|a|\leq\omega} (-1)^{|a|} |w_a\rangle\langle\partial^a\delta|$$

be a projector on $\mathcal{D}_\omega(\mathbb{R}^4)$. The representation of the Lorentz group on $\mathcal{D}_\omega(\mathbb{R}^4)^\perp = \{\sum_{|a|\leq 2} C_a \partial^a\delta | C_a\} \subset \mathcal{D}'(\mathbb{R}^4)$ has the irreducible sub-representations

$$(j_1, j_2) = (0,0), (\frac{1}{2}, \frac{1}{2}), (1,1)$$

with the eigenvalues $c = 0, 3, 8$ of the Casimir operator C_0 (D.25). The latter reads

$$\begin{aligned} C_0 &= -\frac{1}{2}(x_\mu\partial_\nu - x_\nu\partial_\mu)(x^\mu\partial^\nu - x^\nu\partial^\mu) \\ &= (x^\mu\partial_\mu)^2 + 2x^\mu\partial_\mu - x^2\Box \end{aligned}$$

by using (D.19)-(D.20). Following (D.16) and (D.17) a Lorentz invariant extension of t_ω is obtained by

$$t_{\text{inv}} := P(t_\omega \circ W) \quad \text{with} \quad P := (1 - \frac{1}{3}C_0)(1 - \frac{1}{8}C_0) \ .$$

E Time ordered products

One can define time ordered products ('T-products') $(T_n)_{n\in\mathbb{N}}$ by a direct translation of the axioms for retarded products (given in Sect. 2) with the following modifications:

- T_n is required to be symmetrical in *all* factors.
- Causality is expressed by causal factorization:

$$\begin{gathered} T(A_1(x_1), ..., A_n(x_n)) = \\ T(A_1(x_1), ..., A_k(x_k)) \star T(A_{k+1}(x_{k+1}), ..., A_n(x_n)) \end{gathered} \tag{E.1}$$

  if $\{x_1, ..., x_k\} \cap (\{x_{k+1}, ..., x_n\} + \bar{V}_-) = \emptyset$.

- Among the defining properties of T-products there is none corresponding to the GLZ relation (2.27). The information corresponding to (2.27) is (1.1)-(1.2), i.e. the definition of retarded products as coefficients of the interacting fields which are obtained from the T-products by Bogoliubov's formula. (In Proposition 2 of [12] we start from the T-products and derive the GLZ relation (2.27). This derivation uses only (1.1)-(1.2), and linearity and Symmetry are needed in order that (1.2) determines $R_{n,1}$ also for non-diagonal entries.)

The generating functional of the T-products is the (local) S-matrix $\mathbf{S}(F) \stackrel{\mathrm{def}}{=} T(e_\otimes^{iF})$, $F \in \mathcal{F}_{\mathrm{loc}}$.

These defining properties of T-products are **equivalent** to our defining properties of retarded products in the sense of the *unique* correspondence

$$(T_n)_{n\in\{1,2,...,N+1\}} \longleftrightarrow (R_{n,1})_{n\in\{0,1,...,N\}} \tag{E.2}$$

given by (1.1)-(1.2). Using the anti-chronological products $(\bar{T}_n)_{n\in\mathbb{N}}$, which are defined[28] by $\bar{T}(e_\otimes^{-iF}) \stackrel{\mathrm{def}}{=} \mathbf{S}(F)^{-1}$, the correspondence (E.2) can be written more explicitly:

$$R_{n,1}(F_1 \otimes ... \otimes F_n; F) \stackrel{\mathrm{def}}{=} i^n \sum_{I\subset\{1,...,n\}} (-1)^{|I|} \bar{T}_{|I|}(\otimes_{l\in I} F_l) T_{|I^c|+1}((\otimes_{j\in I^c} F_j) \otimes F) \; . \tag{E.3}$$

This formula can also be used to construct inductively the T-products from the retarded products: it yields T_{n+1} in terms of $R_{n,1}$ and $T_l, \bar{T}_k$ with $k, l \leq n$. Alternatively, to obtain a direct formula for T_n in terms of the $\{R_{l,1} | 0 \leq l \leq n-1\}$, we write Bogoliubov's formula (1.1) in the form

$$F_{\lambda F} = -i\mathbf{S}(\lambda F)^{-1} \frac{d}{d\lambda} \mathbf{S}(\lambda F) \; . \tag{E.4}$$

This differential equation is solved by the Dyson series

$$\mathbf{S}(\lambda F) = \mathbf{1} + \sum_{k=1}^{\infty} i^k \int_0^{\lambda} d\lambda_k \int_0^{\lambda_k} d\lambda_{k-1} ... \int_0^{\lambda_2} d\lambda_1 \; F_{\lambda_1 F} ... F_{\lambda_{k-1} F} F_{\lambda_k F} \; . \tag{E.5}$$

[28] Note that $\overline{T}_m$ is uniquely determined in terms of the T-products T_l, $1 \leq l \leq m$.

For $\lambda = 1$ the term of n-th order in F reads

$$T_n(F^{\otimes n}) = \sum_{k=1}^{n} i^{k-n} \sum_{l_1+...+l_k=n-k} \frac{n!}{l_1!l_2!...l_k!}$$
$$\cdot \frac{1}{(l_1+1)(l_1+l_2+2)...(l_1+l_2+...+l_k+k)} R_{l_1,1}(F^{\otimes l_1}, F)...R_{l_k,1}(F^{\otimes l_k}, F) \ , \tag{E.6}$$

where we have used $R_{l,1}((\lambda F)^{\otimes l}, F) = \lambda^l R_{l,1}(F^{\otimes l}, F)$ and computed the λ-integrals. (E.6) agrees with formula[29] (55) of [20], which is derived there in a different way.

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